

# More General Transfer Function Models

- Poles and Zeros:
- The dynamic behavior of a transfer function model can be characterized by the numerical value of its poles and zeros.
- General Representation of ATF:

There are two equivalent representations:

$$G(s) = \frac{\sum_{i=0}^m b_i s^i}{\sum_{i=0}^n a_i s^i} \quad (4-40)$$

$$G(s) = \frac{b_m (s - z_1)(s - z_2) \dots (s - z_m)}{a_n (s - p_1)(s - p_2) \dots (s - p_n)} \quad (6-7)$$

where  $\{z_i\}$  are the “zeros” and  $\{p_i\}$  are the “poles”.

- We will assume that there are no “pole-zero” calculations. That is, that no pole has the same numerical value as a zero.
- *Review:*  $n \geq m$  in order to have a physically realizable system.

## Example 6.2

For the case of a single zero in an overdamped second-order transfer function,

$$G(s) = \frac{K(\tau_a s + 1)}{(\tau_1 s + 1)(\tau_2 s + 1)} \quad (6-14)$$

calculate the response to the step input of magnitude  $M$  and plot the results qualitatively.

### Solution

The response of this system to a step change in input is

$$y(t) = KM \left( 1 + \frac{\tau_a - \tau_1}{\tau_1 - \tau_2} e^{-t/\tau_1} + \frac{\tau_a - \tau_2}{\tau_2 - \tau_1} e^{-t/\tau_2} \right) \quad (6-15)$$

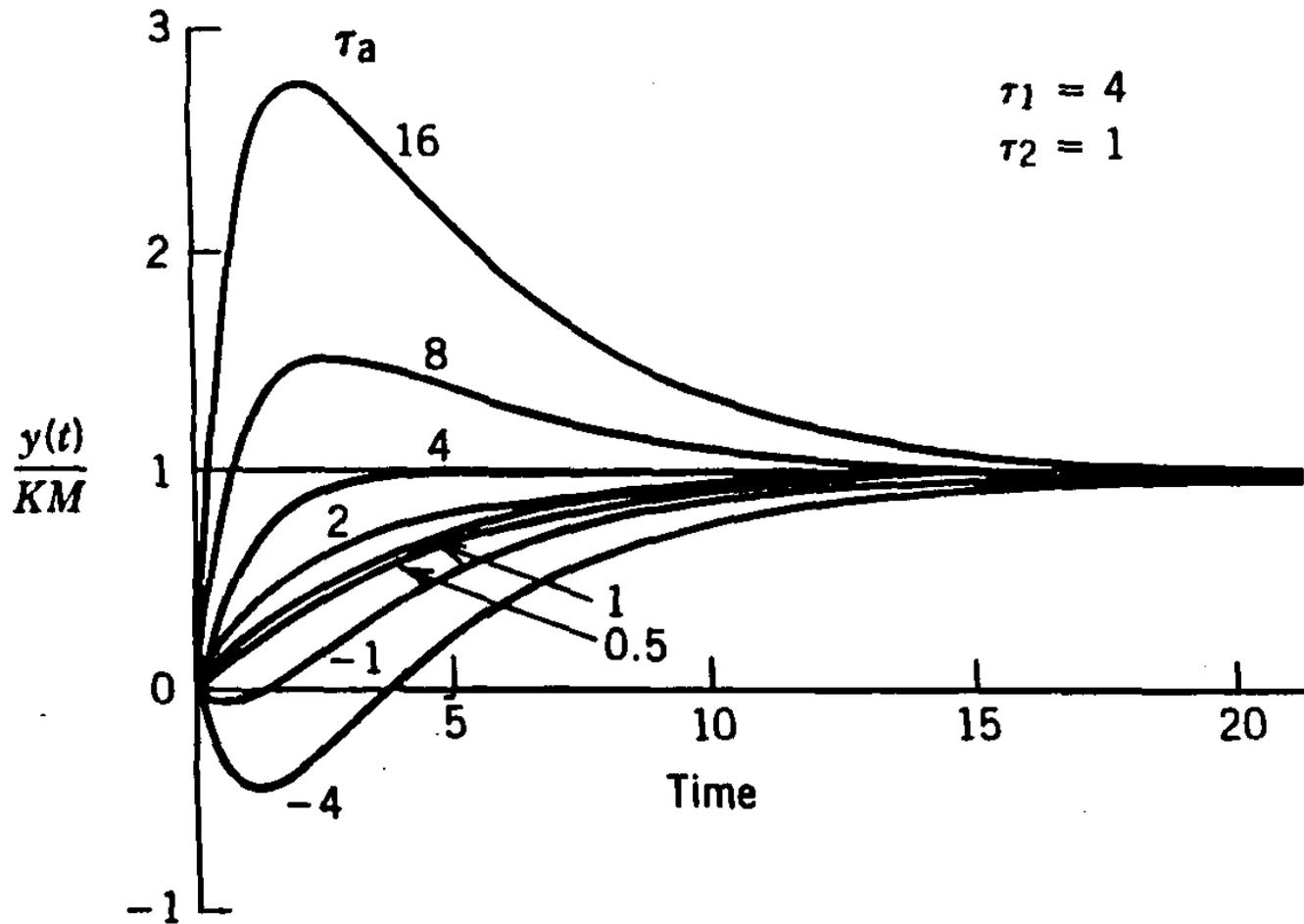
Note that  $y(t \rightarrow \infty) = KM$  as expected; hence, the effect of including the single zero does not change the final value nor does it change the number or location of the response modes. But the zero does affect how the response modes (exponential terms) are weighted in the solution, Eq. 6-15.

A certain amount of mathematical analysis (see Exercises 6.4, 6.5, and 6.6) will show that there are three types of responses involved here:

**Case a:**  $\tau_a > \tau_1$

**Case b:**  $0 < \tau_a \leq \tau_1$

**Case c:**  $\tau_a < 0$



**Figure 6.3.** Step response of an overdamped second-order system (Eq. 6-14) with a single zero.

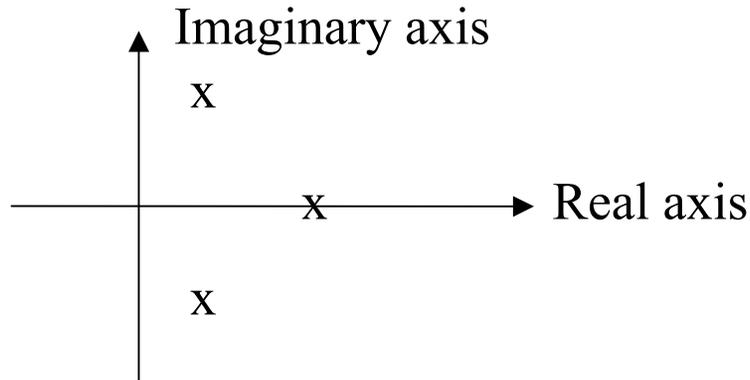
# Summary: Effects of Pole and Zero Locations

## 1. Poles

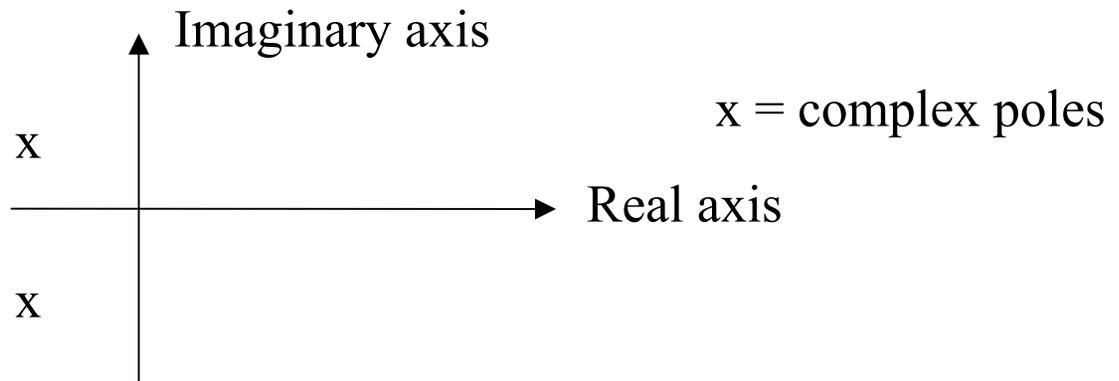
- Pole in “right half plane (RHP)”: results in unstable system (i.e., unstable step responses)

$$p = a + bj$$

$$(j = \sqrt{-1})$$



- Complex pole: results in oscillatory responses

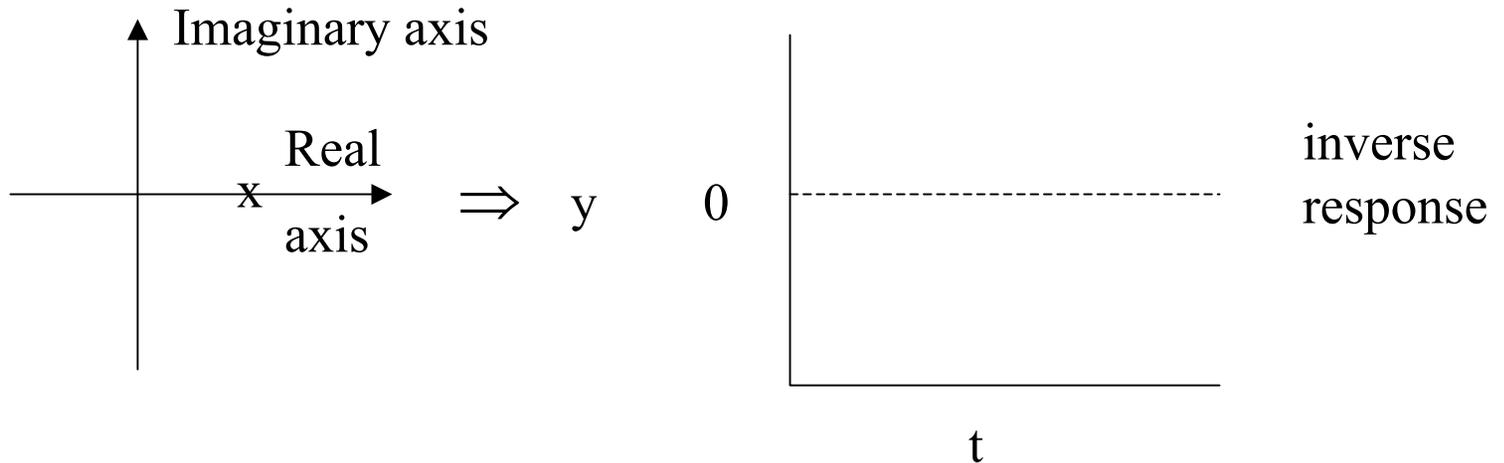


- Pole at the origin ( $1/s$  term in TF model): results in an “integrating process”

## 2. Zeros

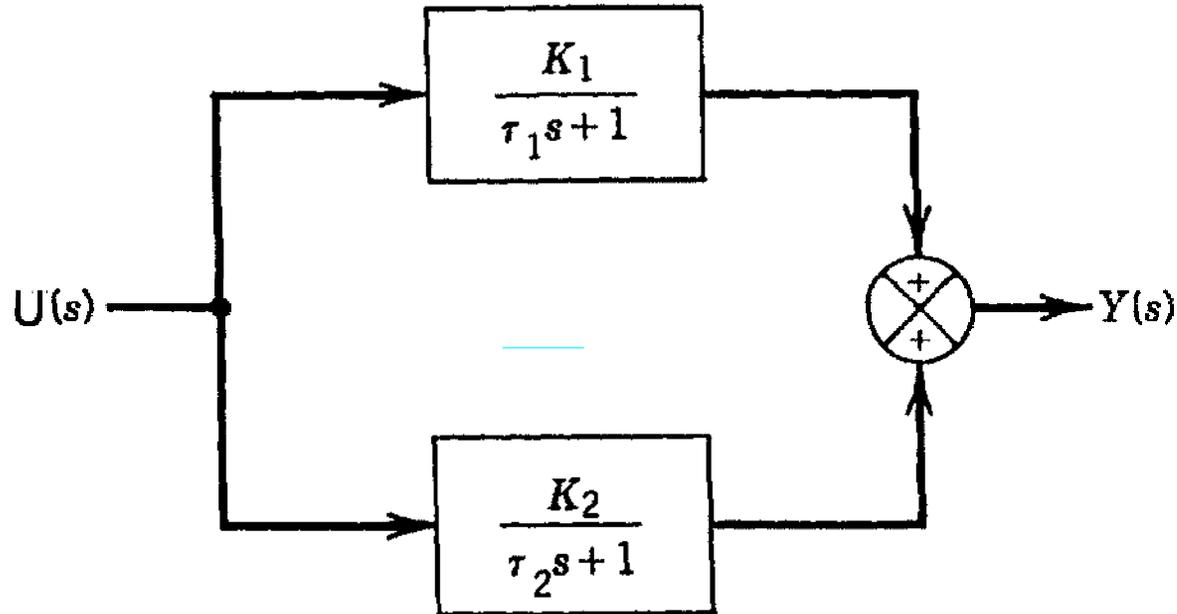
Note: Zeros have no effect on system stability.

- *Zero in RHP*: results in an inverse response to a step change in the input



- *Zero in left half plane*: may result in “overshoot” during a step response (see Fig. 6.3).

# Inverse Response Due to Two Competing Effects



**Figure 6.4.** Two first-order process elements acting in parallel.

An inverse response occurs if:

$$-\frac{K_2}{K_1} > \frac{\tau_2}{\tau_1} \quad (6-22)$$

# Time Delays

Time delays occur due to:

1. Fluid flow in a pipe
2. Transport of solid material (e.g., conveyor belt)
3. Chemical analysis
  - Sampling line delay
  - Time required to do the analysis (e.g., on-line gas chromatograph)

## Mathematical description:

A time delay,  $\theta$ , between an input  $u$  and an output  $y$  results in the following expression:

$$y(t) = \begin{cases} 0 & \text{for } t < \theta \\ u(t - \theta) & \text{for } t \geq \theta \end{cases} \quad (6-27)$$

## *Example: Turbulent flow in a pipe*

Let,  $u \triangleq$  fluid property (e.g., temperature or composition) at point 1

$y \triangleq$  fluid property at point 2

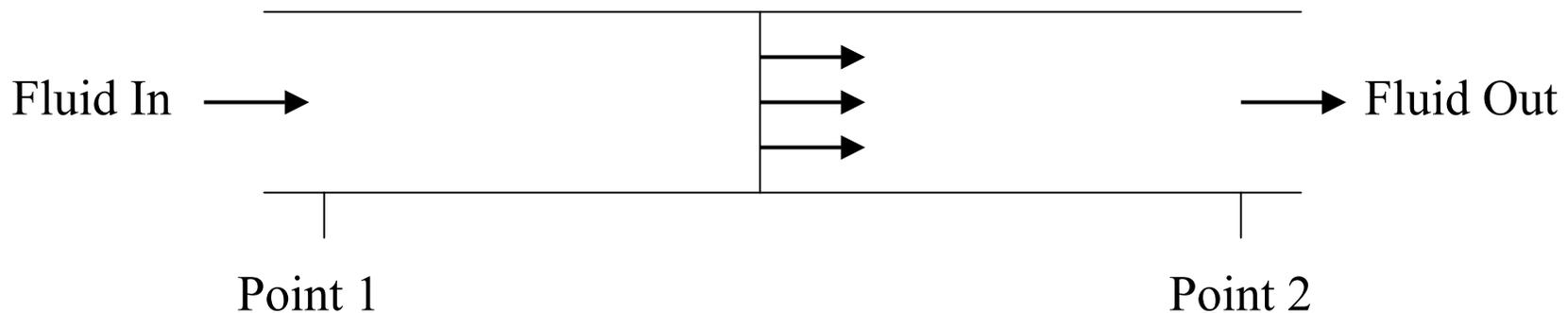
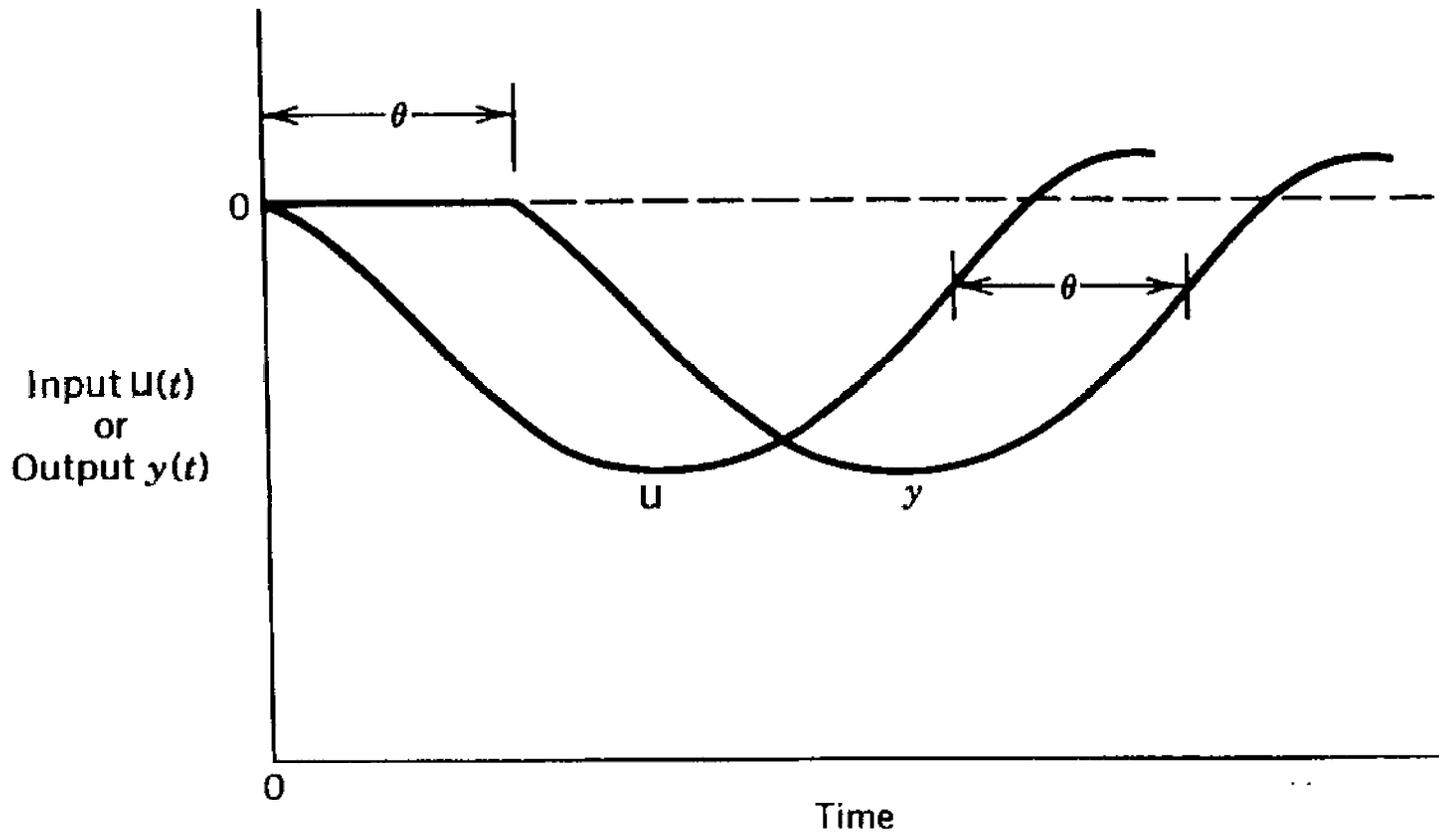


Figure 6.5

Assume that the velocity profile is “flat”, that is, the velocity is uniform over the cross-sectional area. This situation is analyzed in Example 6.5 and Fig. 6.6.



**Figure 6.6.** The effect of a pure time delay is a translation of the function in time.

### *Example 6.5*

For the pipe section illustrated in Fig. 6.5, find the transfer functions:

- (a) relating the mass flow rate of liquid at 2,  $w_2$ , to the mass flow rate of liquid at 1,  $w_1$ ,
- (b) relating the concentration of a chemical species at 2 to the concentration at 1. Assume that the liquid is incompressible.

#### **Solution**

(a) First we make an overall material balance on the pipe segment in question. Since there can be no accumulation (incompressible fluid),

$$\text{material in} = \text{material out} \Rightarrow w_1(t) = w_2(t)$$

Putting (6-30) in deviation form and taking Laplace transforms yields the transfer function,

$$\frac{W_2'(s)}{W_1'(s)} = 1$$

(b) Observing a very small cell of material passing point 1 at time  $t$ , we note that it contains  $Vc_1(t)$  units of the chemical species of interest where  $V$  is the total volume of material in the cell. If, at time  $t + \theta$ , the cell passes point 2, it contains  $Vc_2(t + \theta)$  units of the species. If the material moves in plug flow, not mixing at all with adjacent material, then the amount of species in the cell is constant:

$$Vc_2(t + \theta) = Vc_1(t) \quad (6-30)$$

or

$$c_2(t + \theta) = c_1(t) \quad (6-31)$$

An equivalent way of writing (6-31) is

$$c_2(t) = c_1(t - \theta) \quad (6-32)$$

if the flow rate is constant. Putting (6-32) in deviation form and taking Laplace transforms yields

$$\frac{C_2'(s)}{C_1'(s)} = e^{-\theta s} \quad (6-33)$$

## Time Delays (continued)

Transfer Function Representation:

$$\frac{Y(s)}{U(s)} = e^{-\theta s} \quad (6-28)$$

Note that  $\theta$  has units of time (e.g., minutes, hours)

## Polynomial Approximations to $e^{-\theta s}$ :

For purposes of analysis using analytical solutions to transfer functions, polynomial approximations for  $e^{-\theta s}$  are commonly used. Example: simulation software such as MATLAB and MatrixX.

Two widely used approximations are:

1. *Taylor Series Expansion:*

$$e^{-\theta s} = 1 - \theta s + \frac{\theta^2 s^2}{2!} - \frac{\theta^3 s^3}{3!} + \frac{\theta^4 s^4}{4!} + \dots \quad (6-34)$$

The approximation is obtained by truncating after only a few terms.

## 2. *Padé Approximations:*

Many are available. For example, the 1/1 approximation is,

$$e^{-\theta s} \approx \frac{1 - \frac{\theta}{2}s}{1 + \frac{\theta}{2}s} \quad (6-35)$$

### *Implications for Control:*

Time delays are very bad for control because they involve a delay of information.

# Interacting vs. Noninteracting Systems

- Consider a process with several invariables and several output variables. The process is said to be *interacting* if:
  - Each input affects more than one output.
  - or
  - A change in one output affects the other outputs.

Otherwise, the process is called *noninteracting*.

- As an example, we will consider the two liquid-level storage systems shown in Figs. 4.3 and 6.13.
- In general, transfer functions for interacting processes are more complicated than those for noninteracting processes.

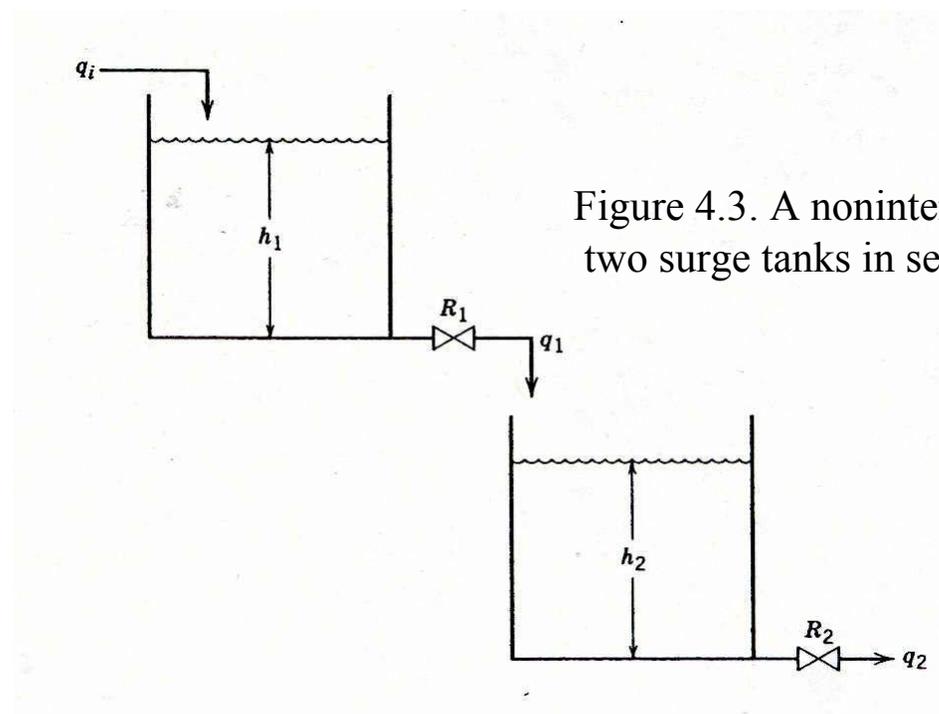


Figure 4.3. A noninteracting system:  
two surge tanks in series.

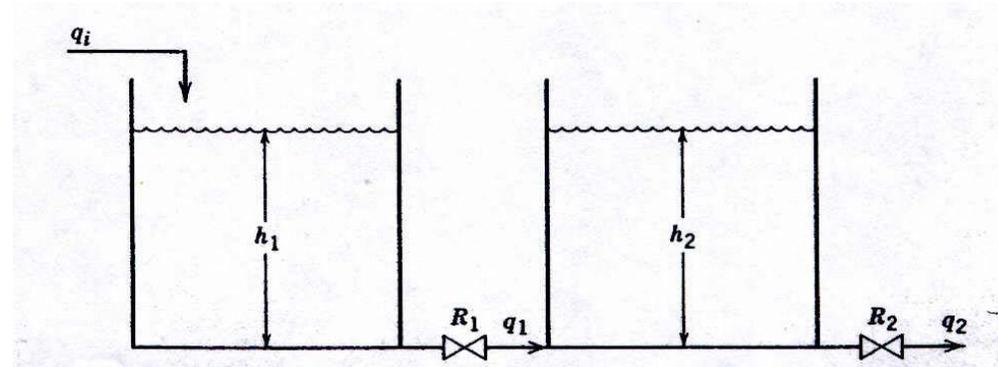


Figure 6.13. Two tanks in series whose liquid levels interact.

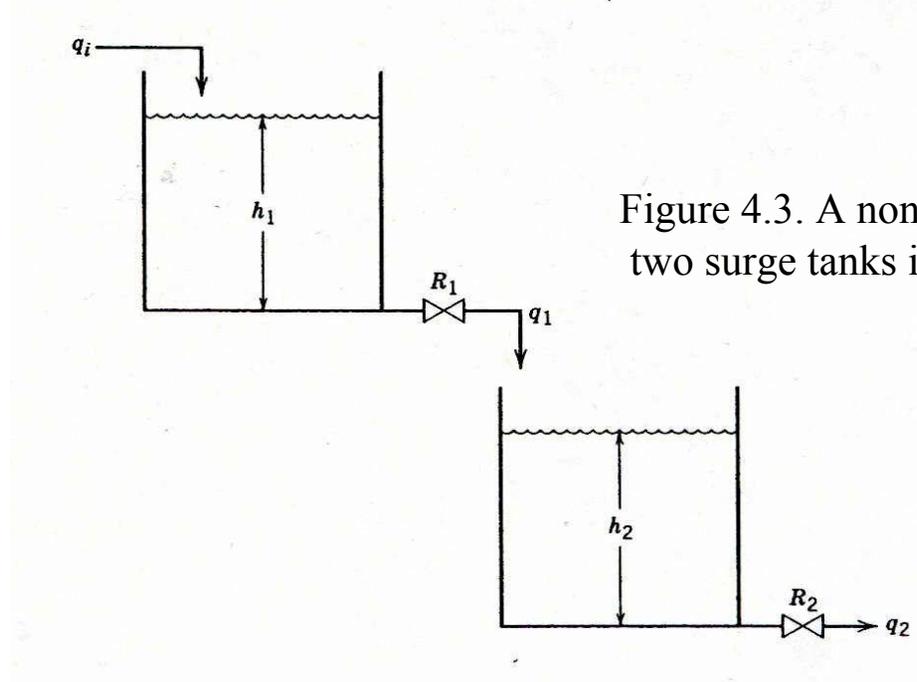


Figure 4.3. A noninteracting system: two surge tanks in series.

**Mass Balance:** 
$$A_1 \frac{dh_1}{dt} = q_i - q_1 \quad (4-48)$$

**Valve Relation:** 
$$q_1 = \frac{1}{R_1} h_1 \quad (4-49)$$

Substituting (4-49) into (4-48) eliminates  $q_1$ :

$$A_1 \frac{dh_1}{dt} = q_i - \frac{1}{R_1} h_1 \quad (4-50)$$

Putting (4-49) and (4-50) into deviation variable form gives

$$A_1 \frac{dh_1'}{dt} = q_i' - \frac{1}{R_1} h_1' \quad (4-51)$$

$$q_1' = \frac{1}{R_1} h_1' \quad (4-52)$$

The transfer function relating  $H_1'(s)$  to  $Q_{1i}'(s)$  is found by transforming (4-51) and rearranging to obtain

$$\frac{H_1'(s)}{Q_i'(s)} = \frac{R_1}{A_1 R_1 s + 1} = \frac{K_1}{\tau_1 s + 1} \quad (4-53)$$

where  $K_1 \triangleq R_1$  and  $\tau_1 \triangleq A_1 R_1$ . Similarly, the transfer function relating  $Q_1'(s)$  to  $H_1'(s)$  is obtained by transforming (4-52).

$$\frac{Q'_1(s)}{H'_1(s)} = \frac{1}{R_1} = \frac{1}{K_1} \quad (4-54)$$

The same procedure leads to the corresponding transfer functions for Tank 2,

$$\frac{H'_2(s)}{Q'_2(s)} = \frac{R_2}{A_2 R_2 s + 1} = \frac{K_2}{\tau_2 s + 1} \quad (4-55)$$

$$\frac{Q'_2(s)}{H'_2(s)} = \frac{1}{R_2} = \frac{1}{K_2} \quad (4-56)$$

where  $K_2 \triangleq R_2$  and  $\tau_2 \triangleq A_2 R_2$ . Note that the desired transfer function relating the outflow from Tank 2 to the inflow to Tank 1 can be derived by forming the product of (4-53) through (4-56).

$$\frac{Q'_2(s)}{Q'_i(s)} = \frac{Q'_2(s)}{H'_2(s)} \frac{H'_2(s)}{Q'_1(s)} \frac{Q'_1(s)}{H'_1(s)} \frac{H'_1(s)}{Q'_i(s)} \quad (4-57)$$

or

$$\frac{Q'_2(s)}{Q'_i(s)} = \frac{1}{K_2} \frac{K_2}{\tau_2 s + 1} \frac{1}{K_1} \frac{K_1}{\tau_1 s + 1} \quad (4-58)$$

which can be simplified to yield

$$\frac{Q'_2(s)}{Q'_i(s)} = \frac{1}{(\tau_1 s + 1)(\tau_2 s + 1)} \quad (4-59)$$

a second-order transfer function (does unity gain make sense on physical grounds?). Figure 4.4 is a block diagram showing *information flow* for this system.

# Block Diagram for Noninteracting Surge Tank System

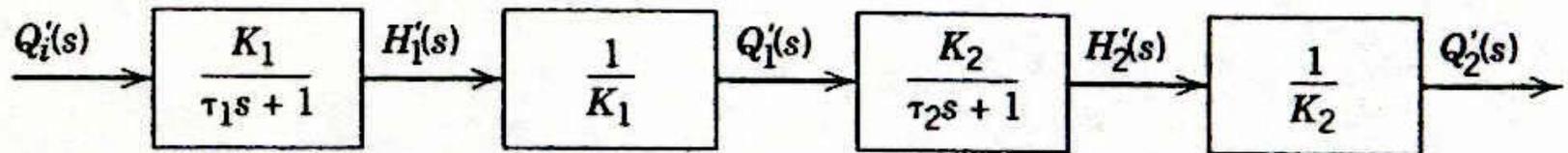


Figure 4.4. Input-output model for two liquid surge tanks in series.

# Dynamic Model of An Interacting Process

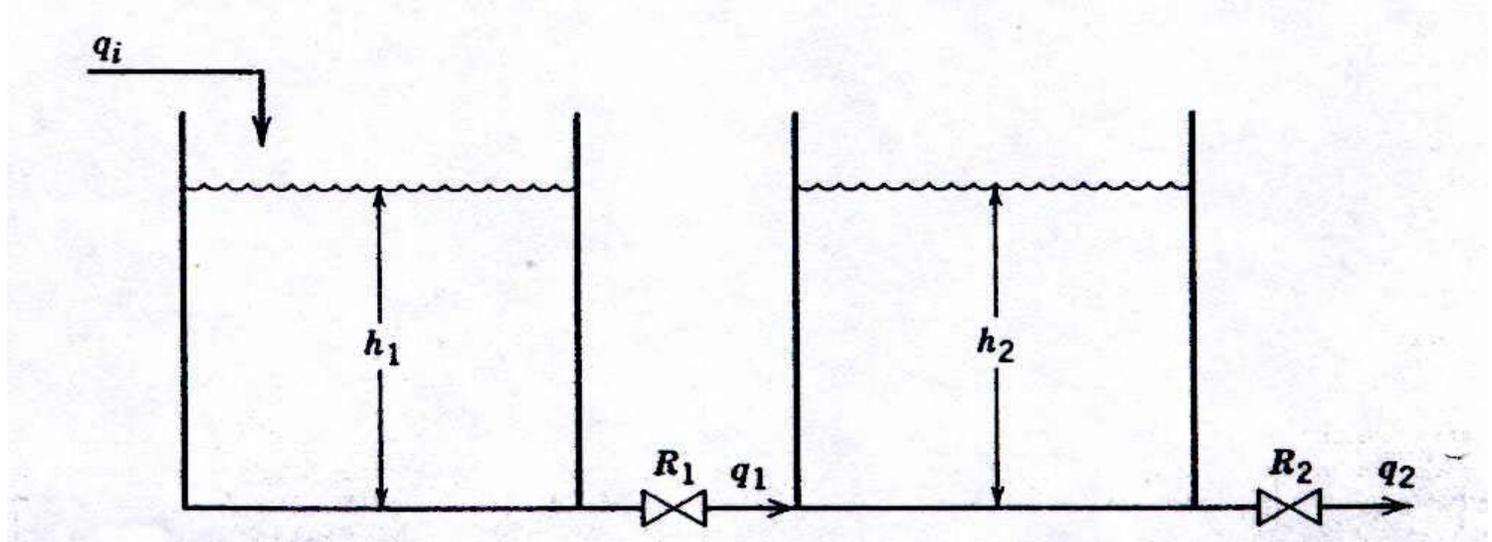


Figure 6.13. Two tanks in series whose liquid levels interact.

$$q_1 = \frac{1}{R_1} (h_1 - h_2) \quad (6-70)$$

The transfer functions for the interacting system are:

$$\frac{H'_2(s)}{Q'_i(s)} = \frac{R_2}{\tau^2 s^2 + 2\zeta\tau s + 1} \quad (6-74)$$

$$\frac{Q'_2(s)}{Q'_i(s)} = \frac{1}{\tau^2 s^2 + 2\zeta\tau s + 1}$$

$$\frac{H'_1(s)}{Q'_i(s)} = \frac{K'_1(\tau_a s + 1)}{\tau^2 s^2 + 2\zeta\tau s + 1} \quad (6-72)$$

where

$$\tau = \sqrt{\tau_1 \tau_2}, \quad \zeta \triangleq \frac{\tau_1 + \tau_2 + R_2 A_1}{2\sqrt{\tau_1 \tau_2}}, \quad \text{and} \quad \tau_a \triangleq R_1 R_2 A_2 / (R_1 + R_2)$$

In Exercise 6.15, the reader can show that  $\zeta > 1$  by analyzing the denominator of (6-71); hence, the transfer function is overdamped, second order, and has a negative zero.

# Model Comparison

- **Noninteracting system**

$$\frac{Q'_2(s)}{Q'_i(s)} = \frac{1}{(\tau_1 s + 1)(\tau_2 s + 1)} \quad (4-59)$$

where  $\tau_1 \triangleq A_1 R_1$  and  $\tau_2 \triangleq A_2 R_2$ .

- **Interacting system**

$$\frac{Q'_2(s)}{Q'_i(s)} = \frac{1}{\tau^2 s^2 + 2\zeta\tau s + 1}$$

where  $\zeta > 1$  and  $\tau \triangleq \sqrt{\tau_1 \tau_2}$

- **General Conclusions**

1. The interacting system has a slower response.  
(Example: consider the special case where  $\tau = \tau_1 = \tau_2$ .)
2. Which two-tank system provides the best damping of inlet flow disturbances?

# Approximation of Higher-Order Transfer Functions

In this section, we present a general approach for approximating high-order transfer function models with lower-order models that have similar dynamic and steady-state characteristics.

In Eq. 6-4 we showed that the transfer function for a time delay can be expressed as a Taylor series expansion. For small values of  $s$ ,

$$e^{-\theta_0 s} \approx 1 - \theta_0 s \quad (6-57)$$

- An alternative first-order approximation consists of the transfer function,

$$e^{-\theta_0 s} = \frac{1}{e^{\theta_0 s}} \approx \frac{1}{1 + \theta_0 s} \quad (6-58)$$

where the time constant has a value of  $\theta_0$ .

- Equations 6-57 and 6-58 were derived to approximate time-delay terms.
- However, these expressions can also be used to approximate the pole or zero term on the right-hand side of the equation by the time-delay term on the left side.

## Skogestad's "half rule"

- Skogestad (2002) has proposed a related approximation method for higher-order models that contain multiple time constants.
- He approximates the largest neglected time constant in the following manner.
- One half of its value is added to the existing time delay (if any) and the other half is added to the smallest retained time constant.
- Time constants that are smaller than the "largest neglected time constant" are approximated as time delays using (6-58).

### Example 6.4

Consider a transfer function:

$$G(s) = \frac{K(-0.1s + 1)}{(5s + 1)(3s + 1)(0.5s + 1)} \quad (6-59)$$

Derive an approximate first-order-plus-time-delay model,

$$\tilde{G}(s) = \frac{Ke^{-\theta s}}{\tau s + 1} \quad (6-60)$$

using two methods:

- (a) The Taylor series expansions of Eqs. 6-57 and 6-58.
- (b) Skogestad's half rule

Compare the normalized responses of  $G(s)$  and the approximate models for a unit step input.

**Solution**

- (a) The dominant time constant (5) is retained. Applying the approximations in (6-57) and (6-58) gives:

$$-0.1s + 1 \approx e^{-0.1s} \quad (6-61)$$

and

$$\frac{1}{3s + 1} \approx e^{-3s} \quad \frac{1}{0.5s + 1} \approx e^{-0.5s} \quad (6-62)$$

Substitution into (6-59) gives the Taylor series approximation,  $\tilde{G}_{TS}(s)$ :

$$\tilde{G}_{TS}(s) = \frac{Ke^{-0.1s} e^{-3s} e^{-0.5s}}{5s + 1} = \frac{Ke^{-3.6s}}{5s + 1} \quad (6-63)$$

(b) To use Skogestad's method, we note that the largest neglected time constant in (6-59) has a value of three.

- According to his “half rule”, half of this value is added to the next largest time constant to generate a new time constant  $\tau = 5 + 0.5(3) = 6.5$ .
- The other half provides a new time delay of  $0.5(3) = 1.5$ .
- The approximation of the RHP zero in (6-61) provides an additional time delay of 0.1.
- Approximating the smallest time constant of 0.5 in (6-59) by (6-58) produces an additional time delay of 0.5.
- Thus the total time delay in (6-60) is,

$$\theta = 1.5 + 0.1 + 0.5 = 2.1$$

and  $G(s)$  can be approximated as:

$$\tilde{G}_{Sk}(s) = \frac{Ke^{-2.1s}}{6.5s + 1} \quad (6-64)$$

The normalized step responses for  $G(s)$  and the two approximate models are shown in Fig. 6.10. Skogestad's method provides better agreement with the actual response.

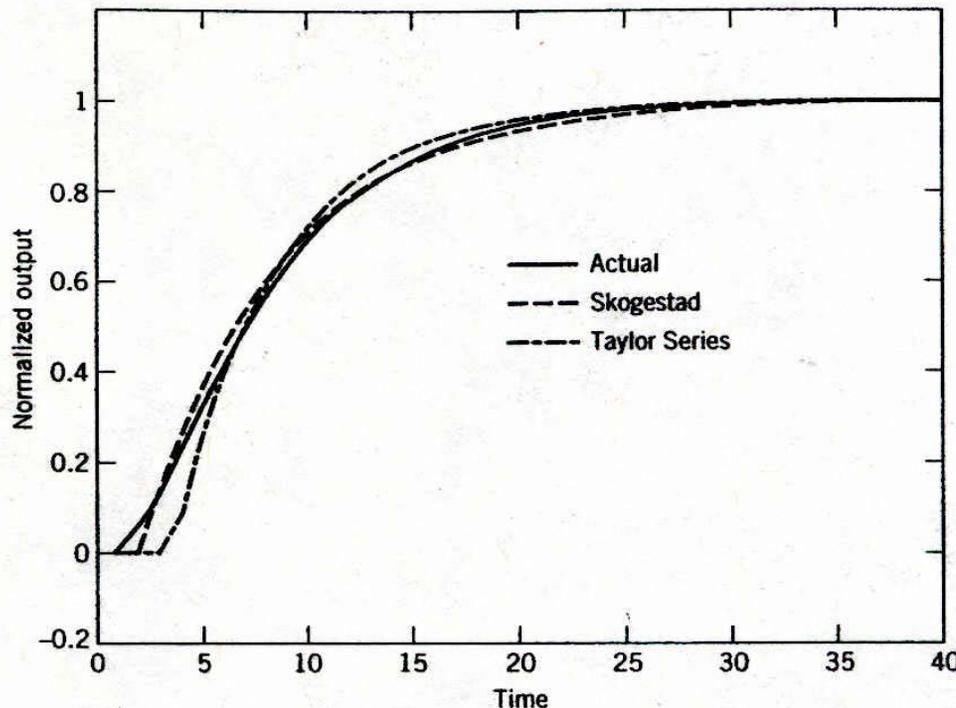


Figure 6.10  
Comparison of the  
actual and  
approximate models  
for Example 6.4.

## Example 6.5

Consider the following transfer function:

$$G(s) = \frac{K(1-s)e^{-s}}{(12s+1)(3s+1)(0.2s+1)(0.05s+1)} \quad (6-65)$$

Use Skogestad's method to derive two approximate models:

- (a) A first-order-plus-time-delay model in the form of (6-60)
- (b) A second-order-plus-time-delay model in the form:

$$\tilde{G}(s) = \frac{Ke^{-\theta s}}{(\tau_1 s + 1)(\tau_2 s + 1)} \quad (6-66)$$

Compare the normalized output responses for  $G(s)$  and the approximate models to a unit step input.

## Solution

- (a) For the first-order-plus-time-delay model, the dominant time constant (12) is retained.
- One-half of the largest neglected time constant (3) is allocated to the retained time constant and one-half to the approximate time delay.
  - Also, the small time constants (0.2 and 0.05) and the zero (1) are added to the original time delay.
  - Thus the model parameters in (6-60) are:

$$\theta = 1 + \frac{3.0}{2} + 0.2 + 0.05 + 1 = 3.75$$

$$\tau = 12 + \frac{3.0}{2} = 13.5$$

(b) An analogous derivation for the second-order-plus-time-delay model gives:

$$\theta = 1 + \frac{0.2}{2} + 0.05 + 1 = 2.15$$

$$\tau_1 = 12, \quad \tau_2 = 3 + 0.1 = 3.1$$

In this case, the half rule is applied to the third largest time constant (0.2). The normalized step responses of the original and approximate transfer functions are shown in Fig. 6.11.

# Multiple-Input, Multiple Output (MIMO) Processes

- Most industrial process control applications involved a number of input (manipulated) and output (controlled) variables.
- These applications often are referred to as multiple-input/multiple-output (MIMO) systems to distinguish them from the simpler single-input/single-output (SISO) systems that have been emphasized so far.
- Modeling MIMO processes is no different conceptually than modeling SISO processes.

- For example, consider the system illustrated in Fig. 6.14.
- Here the level  $h$  in the stirred tank and the temperature  $T$  are to be controlled by adjusting the flow rates of the hot and cold streams  $w_h$  and  $w_c$ , respectively.
- The temperatures of the inlet streams  $T_h$  and  $T_c$  represent potential disturbance variables.
- Note that the outlet flow rate  $w$  is maintained constant and the liquid properties are assumed to be constant in the following derivation.

$$\begin{bmatrix} T'(s) \\ H'(s) \end{bmatrix} = \begin{bmatrix} \frac{(\bar{T}_h - \bar{T})/\bar{w}}{\tau s + 1} & \frac{(\bar{T}_c - \bar{T})/\bar{w}}{\tau s + 1} & \frac{\bar{w}_h/\bar{w}}{\tau s + 1} & \frac{\bar{w}_c/\bar{w}}{\tau s + 1} \\ \frac{1/A\rho}{s} & \frac{1/A\rho}{s} & 0 & 0 \end{bmatrix} \begin{bmatrix} W'_h(s) \\ W'_c(s) \\ T'_h(s) \\ T'_c(s) \end{bmatrix}$$

(6-88)

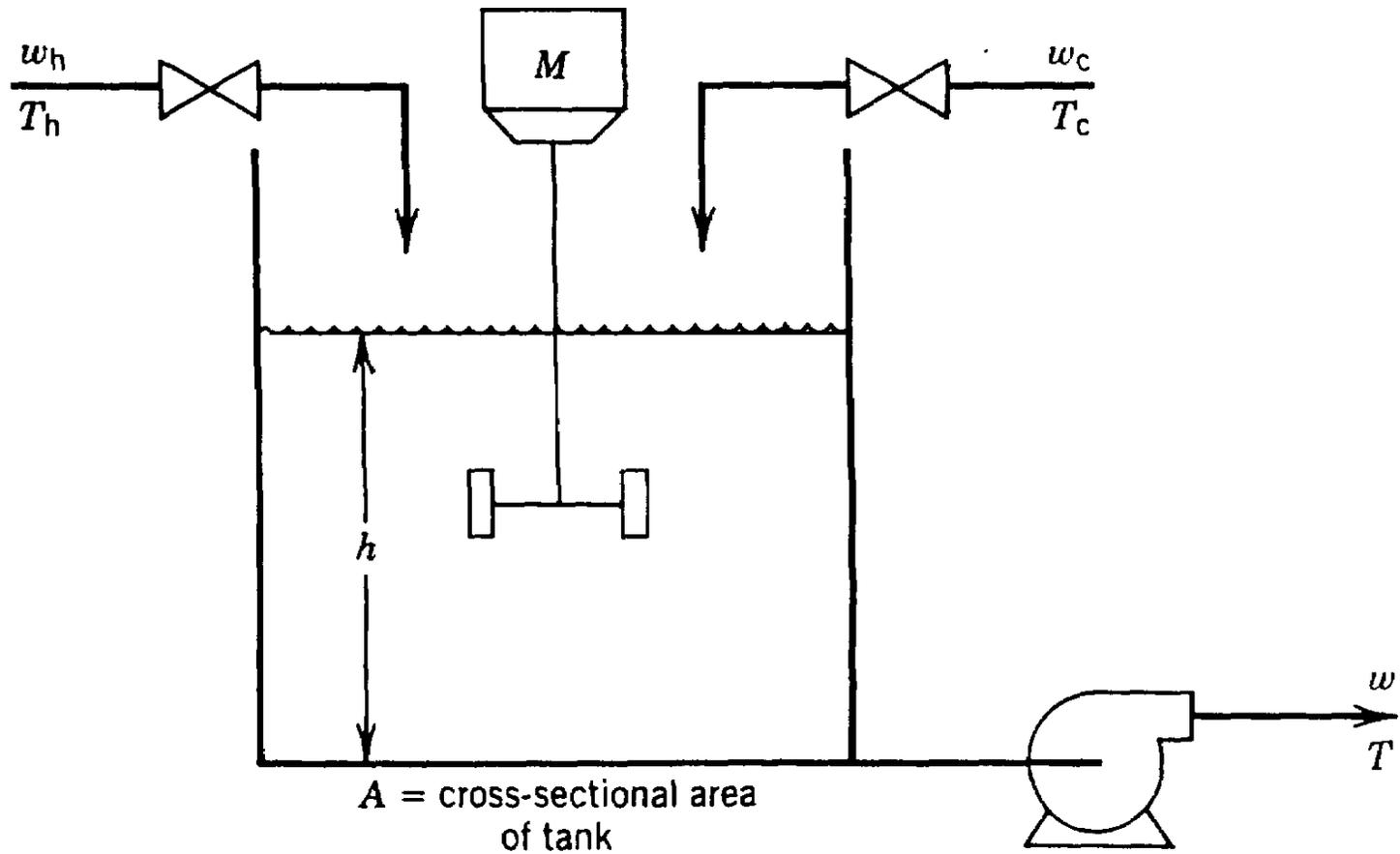


Figure 6.14. A multi-input, multi-output thermal mixing process.

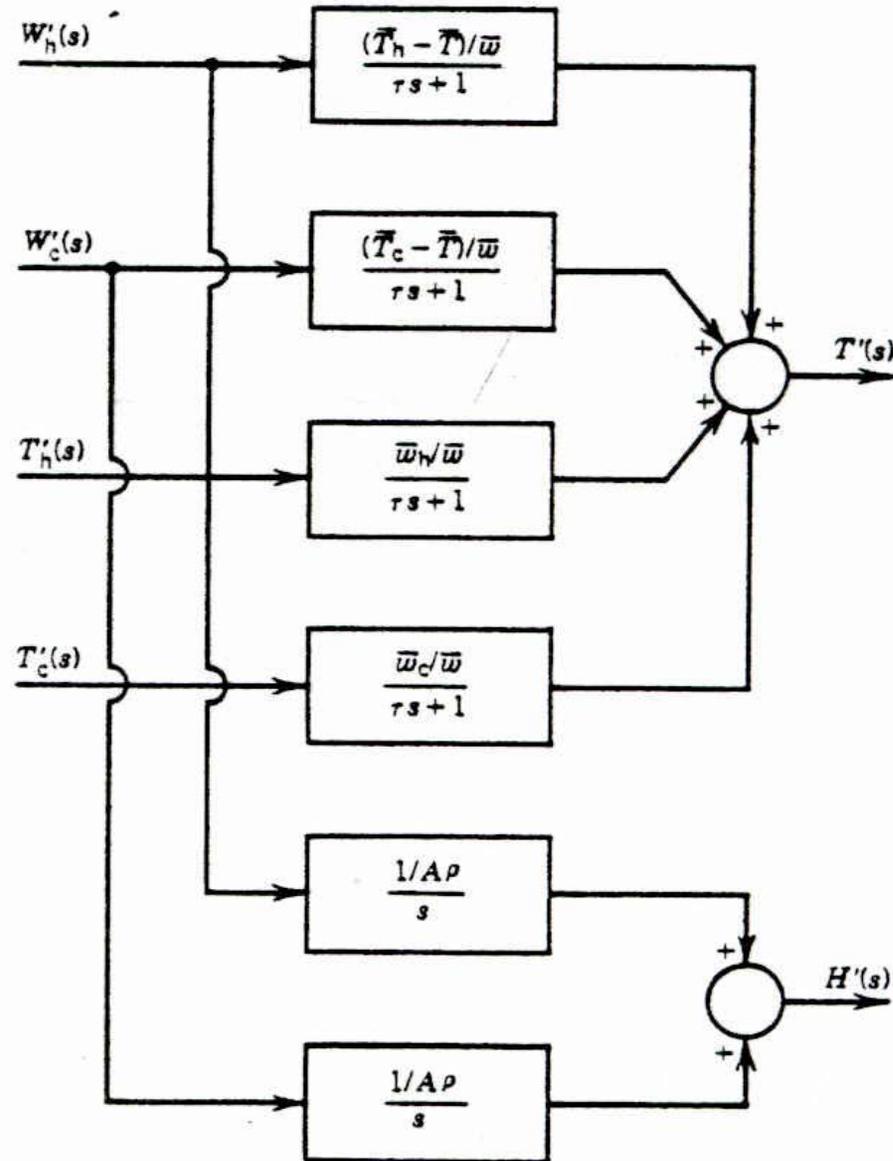


Figure 6.18. Block diagram of the MIMO mixing system with variable level.