Control System Design Based on Frequency Response Analysis

Frequency response concepts and techniques play an important role in control system design and analysis.

Closed-Loop Behavior

In general, a feedback control system should satisfy the following design objectives:

1. Closed-loop stability
2. Good disturbance rejection (without excessive control action)
3. Fast set-point tracking (without excessive control action)
4. A satisfactory degree of robustness to process variations and model uncertainty
5. Low sensitivity to measurement noise

- The block diagram of a general feedback control system is shown in Fig. 14.1.
- It contains three external input signals: set point Y_{sp}, disturbance D, and additive measurement noise, N.

$$Y = \frac{G_d}{1+G_cG}D - \frac{G_cG}{1+G_cG}N + \frac{K_mG_cG_vG_p}{1+G_cG}Y_{sp} \qquad (14\text{-}1)$$

$$E = -\frac{G_dG_m}{1+G_cG}D - \frac{G_m}{1+G_cG}N + \frac{K_m}{1+G_cG}Y_{sp} \qquad (14\text{-}2)$$

$$U = -\frac{G_dG_mG_cG_v}{1+G_cG}D - \frac{G_mG_cG_v}{1+G_cG}N + \frac{K_mG_cG_v}{1+G_cG}Y_{sp} \qquad (14\text{-}3)$$

where $G \triangleq G_vG_pG_m$.

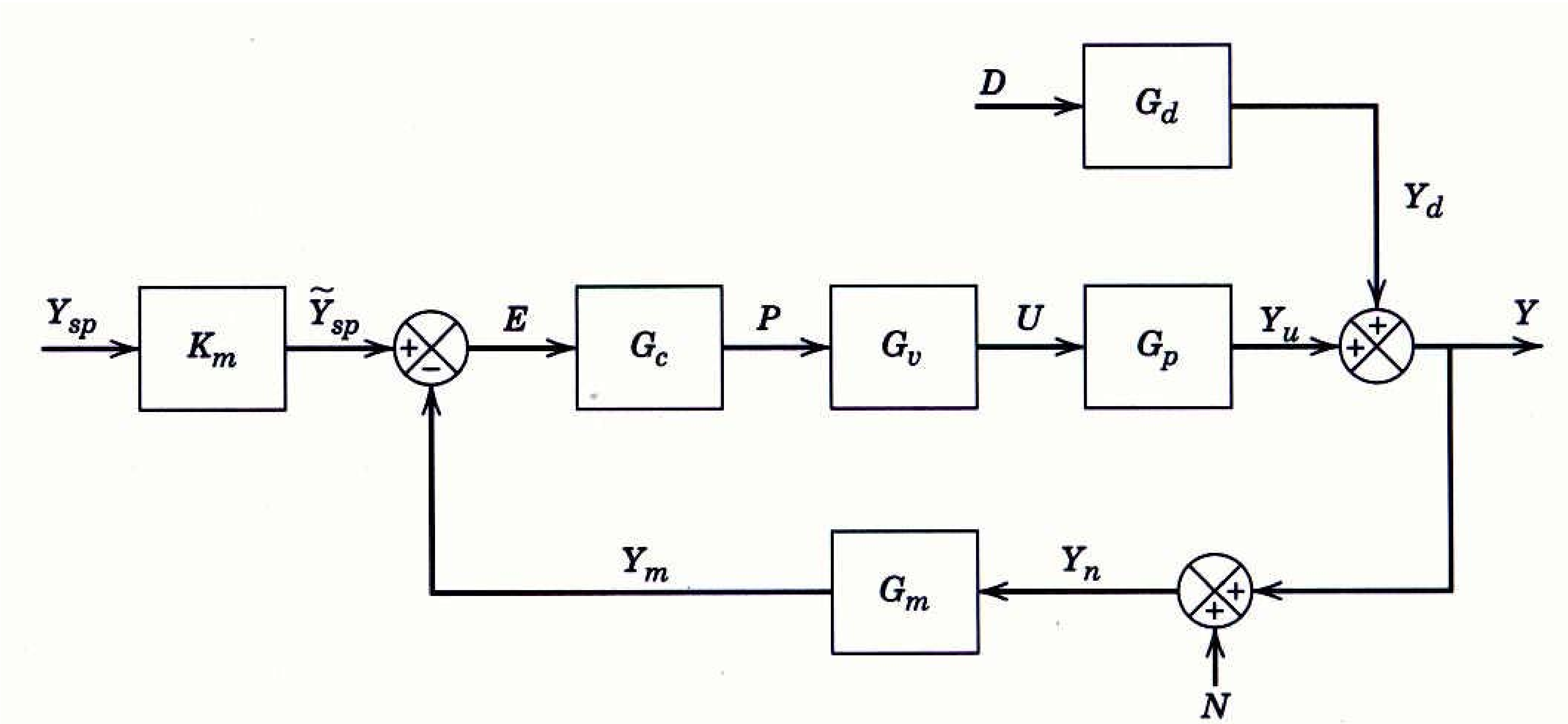


Figure 14.1 Block diagram with a disturbance *D* and measurement noise *N*.

Example 14.1

Consider the feedback system in Fig. 14.1 and the following transfer functions:

$$G_p = G_d = \frac{0.5}{1-2s}, \quad G_v = G_m = 1$$

Suppose that controller G_c is designed to cancel the unstable pole in G_p:

$$G_c = -\frac{3(1-2s)}{s+1}$$

Evaluate closed-loop stability and characterize the output response for a sustained disturbance.

Solution

The characteristic equation, $1 + G_cG = 0$, becomes:

$$1 + \frac{3(1-2s)}{s+1}\frac{0.5}{1-2s} = 0$$

or

$$s + 2.5 = 0$$

In view of the single root at $s = -2.5$, it appears that the closed-loop system is stable. However, if we consider Eq. 14-1 for $N = Y_{sp} = 0$,

$$Y = \frac{G_d}{1+G_cG}D = \frac{-0.5(s+1)}{(1-2s)(s+2.5)}D$$

- This transfer function has an unstable pole at s = +0.5. Thus, the output response to a disturbance is unstable.
- Furthermore, other transfer functions in (14-1) to (14-3) also have unstable poles.
- This apparent contradiction occurs because the characteristic equation does not include all of the information, namely, the unstable pole-zero cancellation.

Example 14.2

Suppose that $G_d = G_p$, $G_m = K_m$ and that G_c is designed so that the closed-loop system is stable and $|GG_c| >> 1$ over the frequency range of interest. Evaluate this control system design strategy for set-point changes, disturbances, and measurement noise. Also consider the behavior of the manipulated variable, U.

Solution

Because $|GG_c| >> 1$,

$$\frac{1}{1+G_cG} \approx 0 \qquad \text{and} \qquad \frac{G_cG}{1+G_cG} \approx 1$$

The first expression and (14-1) suggest that the output response to disturbances will be very good because $Y/D \approx 0$. Next, we consider set-point responses. From Eq. 14-1,

$$\frac{Y}{Y_{sp}} = \frac{K_m G_c G_v G_p}{1+G_cG}$$

Because $G_m = K_m$, $G = G_vG_pK_m$ and the above equation can be written as,

$$\frac{Y}{Y_{sp}} = \frac{G_cG}{1+G_cG}$$

For $|GG_c| >> 1$,

$$\frac{Y}{Y_{sp}} \approx 1$$

Thus, ideal (instantaneous) set-point tracking would occur. Choosing G_c so that $|GG_c| >> 1$ also has an undesirable consequence. The output Y becomes sensitive to noise because $Y \approx -N$ (see the noise term in Eq. 14-1). Thus, a design tradeoff is required.

Bode Stability Criterion

The Bode stability criterion has two important advantages in comparison with the Routh stability criterion of Chapter 11:

1. It provides exact results for processes with time delays, while the Routh stability criterion provides only approximate results due to the polynomial approximation that must be substituted for the time delay.

2. The Bode stability criterion provides a measure of the *relative stability* rather than merely a yes or no answer to the question, "Is the closed-loop system stable?"

Before considering the basis for the Bode stability criterion, it is useful to review the General Stability Criterion of Section 11.1:

A feedback control system is stable if and only if all roots of the characteristic equation lie to the left of the imaginary axis in the complex plane.

Before stating the Bode stability criterion, we need to introduce two important definitions:

1. A *critical frequency* ω_c is defined to be a value of ω for which $\varphi_{OL}(\omega) = -180°$. This frequency is also referred to as a *phase crossover frequency*.

2. A *gain crossover frequency* ω_g is defined to be a value of ω for which $AR_{OL}(\omega) = 1$.

For many control problems, there is only a single ω_c and a single ω_g. But multiple values can occur, as shown in Fig. 14.3 for ω_c.

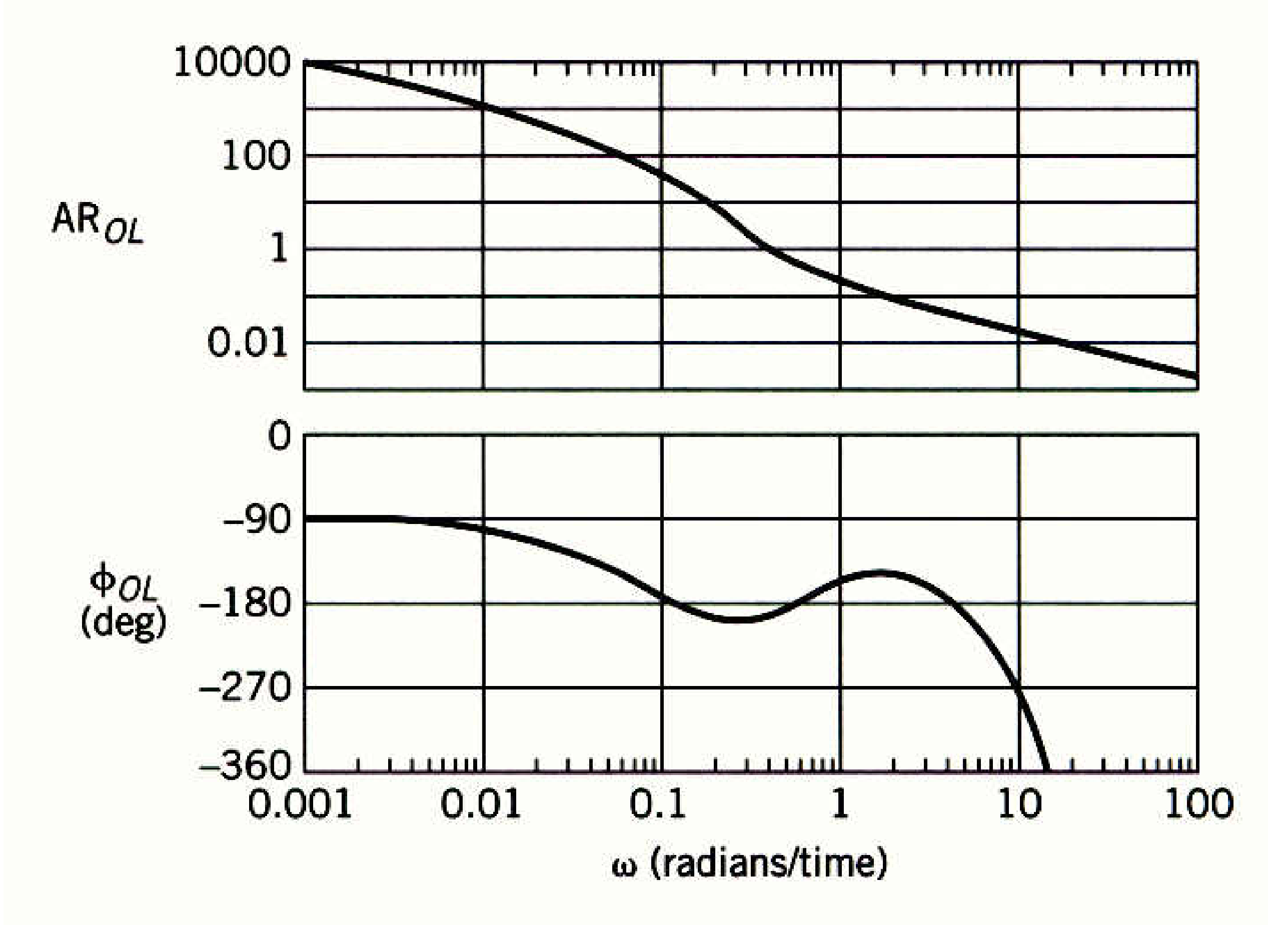


Figure 14.3 Bode plot exhibiting multiple critical frequencies.

Bode Stability Criterion*. Consider an open-loop transfer function $G_{OL}=G_cG_vG_pG_m$ that is strictly proper (more poles than zeros) and has no poles located on or to the right of the imaginary axis, with the possible exception of a single pole at the origin. Assume that the open-loop frequency response has only a single critical frequency ω_c and a single gain crossover frequency ω_g. Then the closed-loop system is stable if $AR_{OL}(\omega_c) < 1$. Otherwise it is unstable.*

Some of the important properties of the Bode stability criterion are:

1. It provides a necessary and sufficient condition for closed-loop stability based on the properties of the open-loop transfer function.

2. Unlike the Routh stability criterion of Chapter 11, the Bode stability criterion is applicable to systems that contain time delays.

3. The Bode stability criterion is very useful for a wide range of process control problems. However, for any $G_{OL}(s)$ that does not satisfy the required conditions, the Nyquist stability criterion of Section 14.3 can be applied.

4. For systems with multiple ω_c or ω_g , the Bode stability criterion has been modified by Hahn et al. (2001) to provide a sufficient condition for stability.

- In order to gain physical insight into why a sustained oscillation occurs at the stability limit, consider the analogy of an adult pushing a child on a swing.

- The child swings in the same arc as long as the adult pushes at the right time, and with the right amount of force.

- Thus the desired "sustained oscillation" places requirements on both timing (that is, *phase*) and applied force (that is, *amplitude*).

- By contrast, if either the force or the timing is not correct, the desired swinging motion ceases, as the child will quickly exclaim.
- A similar requirement occurs when a person bounces a ball.
- To further illustrate why feedback control can produce sustained oscillations, consider the following “thought experiment” for the feedback control system in Figure 14.4. Assume that the open-loop system is stable and that no disturbances occur ($D = 0$).
- Suppose that the set point is varied sinusoidally at the critical frequency, $y_{sp}(t) = A \sin(\omega_c t)$, for a long period of time.
- Assume that during this period the measured output, y_m, is disconnected so that the feedback loop is broken before the comparator.

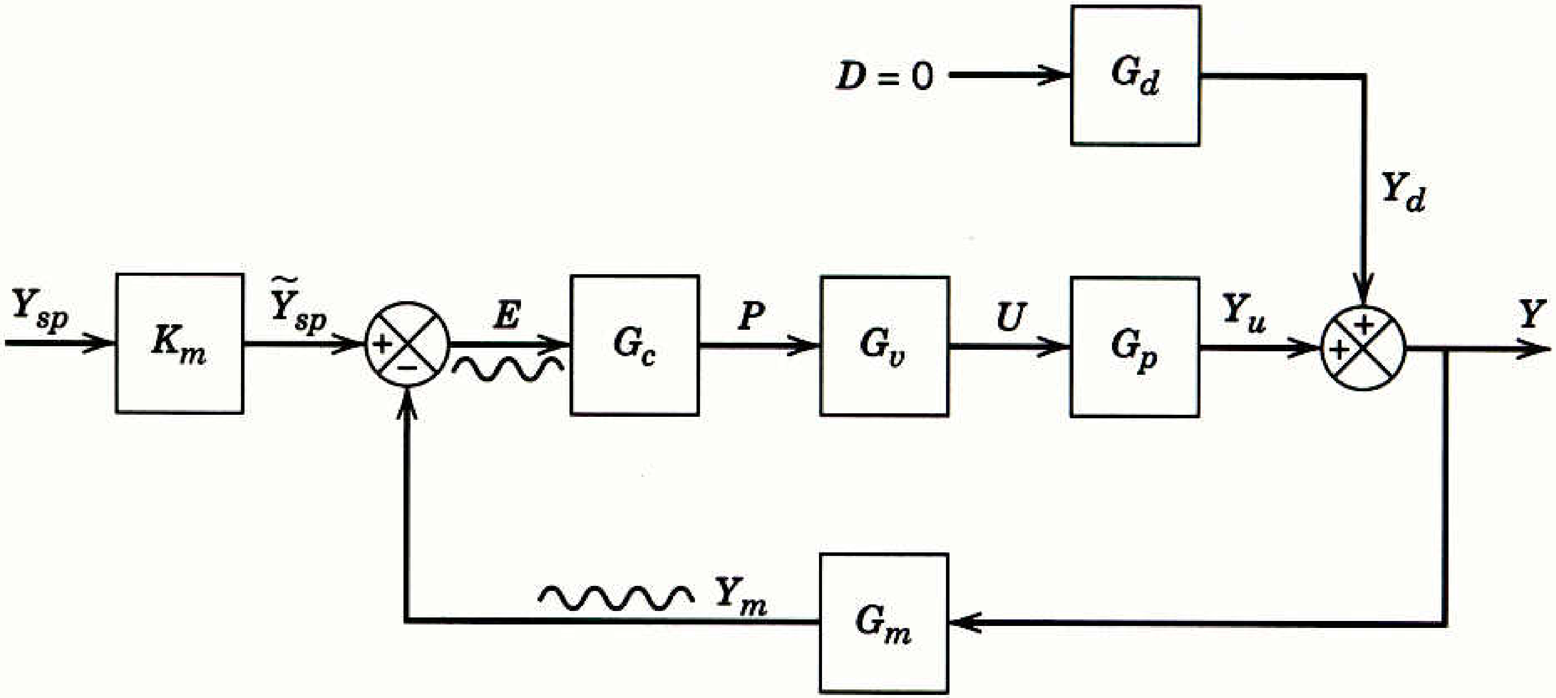


Figure 14.4 Sustained oscillation in a feedback control system.

- After the initial transient dies out, y_m will oscillate at the excitation frequency ω_c because the response of a linear system to a sinusoidal input is a sinusoidal output at the same frequency (see Section 13.2).
- Suppose that two events occur simultaneously: (i) the set point is set to zero and, (ii) y_m is reconnected. If the feedback control system is *marginally stable,* the controlled variable y will then exhibit a sustained sinusoidal oscillation with amplitude A and frequency ω_c.
- To analyze why this special type of oscillation occurs only when $\omega = \omega_c$, note that the sinusoidal signal E in Fig. 14.4 passes through transfer functions G_c, G_v, G_p, and G_m before returning to the comparator.
- In order to have a sustained oscillation after the feedback loop is reconnected, signal Y_m must have the same amplitude as E and a -180° phase shift relative to E.

- Note that the comparator also provides a -180° phase shift due to its negative sign.
- Consequently, after Y_m passes through the comparator, it is in phase with E and has the same amplitude, A.
- Thus, the closed-loop system oscillates indefinitely after the feedback loop is closed because the conditions in Eqs. 14-7 and 14-8 are satisfied.
- But what happens if K_c is increased by a small amount?
- Then, $AR_{OL}(\omega_c)$ is greater than one and the closed-loop system becomes unstable.
- In contrast, if K_c is reduced by a small amount, the oscillation is "damped" and eventually dies out.

Example 14.3

A process has the third-order transfer function (time constant in minutes),

$$G_p(s) = \frac{2}{(0.5s+1)^3}$$

Also, $G_v = 0.1$ and $G_m = 10$. For a proportional controller, evaluate the stability of the closed-loop control system using the Bode stability criterion and three values of K_c: 1, 4, and 20.

Solution

For this example,

$$G_{OL} = G_c G_v G_p G_m = (K_c)(0.1)\frac{2}{(0.5s+1)^3}(10) = \frac{2K_c}{(0.5s+1)^3}$$

Figure 14.5 shows a Bode plot of G_{OL} for three values of K_c. Note that all three cases have the same phase angle plot because the phase lag of a proportional controller is zero for $K_c > 0$.

Next, we consider the amplitude ratio AR_{OL} for each value of K_c. Based on Fig. 14.5, we make the following classifications:

K_c	$AR_{OL}\left(\text{for } \omega = \omega_c\right)$	Classification
1	0.25	Stable
4	1	Marginally stable
20	5	Unstable

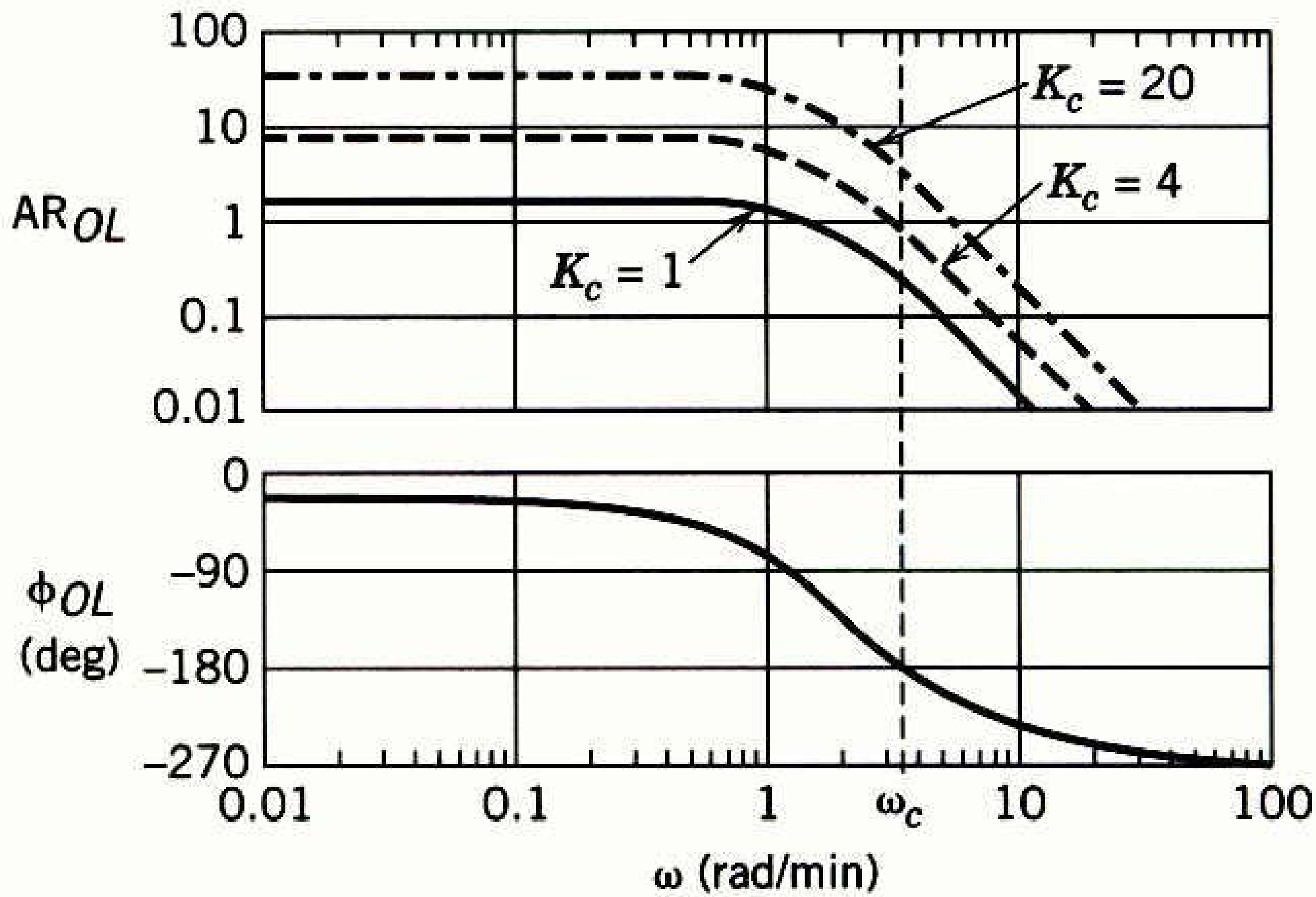


Figure 14.5 Bode plots for $G_{OL} = 2K_c/(0.5s+1)^3$.

In Section 12.5.1 the concept of the ultimate gain was introduced. For proportional-only control, the ultimate gain K_{cu} was defined to be the largest value of K_c that results in a stable closed-loop system. The value of K_{cu} can be determined graphically from a Bode plot for transfer function $G = G_v G_p G_m$. For proportional-only control, $G_{OL} = K_c G$. Because a proportional controller has zero phase lag if $K_c > 0$, ω_c is determined solely by G. Also,

$$AR_{OL}(\omega) = K_c\, AR_G(\omega) \qquad (14\text{-}9)$$

where AR_G denotes the amplitude ratio of G. At the stability limit, $\omega = \omega_c$, $AR_{OL}(\omega_c) = 1$ and $K_c = K_{cu}$. Substituting these expressions into (14-9) and solving for K_{cu} gives an important result:

$$K_{cu} = \frac{1}{AR_G(\omega_c)} \qquad (14\text{-}10)$$

The stability limit for K_c can also be calculated for PI and PID controllers, as demonstrated by Example 14.4.

Nyquist Stability Criterion

- The Nyquist stability criterion is similar to the Bode criterion in that it determines closed-loop stability from the open-loop frequency response characteristics.
- The Nyquist stability criterion is based on two concepts from complex variable theory, *contour mapping* and the *Principle of the Argument*.

Nyquist Stability Criterion. *Consider an open-loop transfer function $G_{OL}(s)$ that is proper and has no unstable pole-zero cancellations. Let N be the number of times that the Nyquist plot for $G_{OL}(s)$ encircles the -1 point in the clockwise direction. Also let P denote the number of poles of $G_{OL}(s)$ that lie to the right of the imaginary axis. Then, $Z = N + P$* where *Z is the number of roots of the characteristic equation that lie to the right of the imaginary axis* (*that is, its number of "zeros"*)*. The closed-loop system is stable if and only if $Z = 0$.*

Some important properties of the Nyquist stability criterion are:

1. It provides a necessary and sufficient condition for closed-loop stability based on the open-loop transfer function.

2. The reason the -1 point is so important can be deduced from the characteristic equation, $1 + G_{OL}(s) = 0$. This equation can also be written as $G_{OL}(s) = -1$, which implies that $AR_{OL} = 1$ and $\varphi_{OL} = -180^\circ$, as noted earlier. The -1 point is referred to as the *critical point*.

3. Most process control problems are open-loop stable. For these situations, $P = 0$ and thus $Z = N$. Consequently, the closed-loop system is unstable if the Nyquist plot for $G_{OL}(s)$ encircles the -1 point, one or more times.

4. A negative value of N indicates that the -1 point is encircled in the opposite direction (counter-clockwise). This situation implies that each countercurrent encirclement can stabilize one unstable pole of the open-loop system.

5. Unlike the Bode stability criterion, the Nyquist stability criterion is applicable to open-loop unstable processes.

6. Unlike the Bode stability criterion, the Nyquist stability criterion can be applied when multiple values of ω_c or ω_g occur (cf. Fig. 14.3).

Example 14.6

Evaluate the stability of the closed-loop system in Fig. 14.1 for:

$$G_p(s) = \frac{4e^{-s}}{5s+1}$$

(the time constants and delay have units of minutes)

$$G_v = 2, \quad G_m = 0.25, \quad G_c = K_c$$

Obtain ω_c and K_{cu} from a Bode plot. Let $K_c = 1.5K_{cu}$ and draw the Nyquist plot for the resulting open-loop system.

Chapter 14

Solution

The Bode plot for G_{OL} and $K_c = 1$ is shown in Figure 14.7. For $\omega_c = 1.69$ rad/min, $\phi_{OL} = -180°$ and $AR_{OL} = 0.235$. For $K_c = 1$, $AR_{OL} = AR_G$ and K_{cu} can be calculated from Eq. 14-10. Thus, $K_{cu} = 1/0.235 = 4.25$. Setting $K_c = 1.5K_{cu}$ gives $K_c = 6.38$.

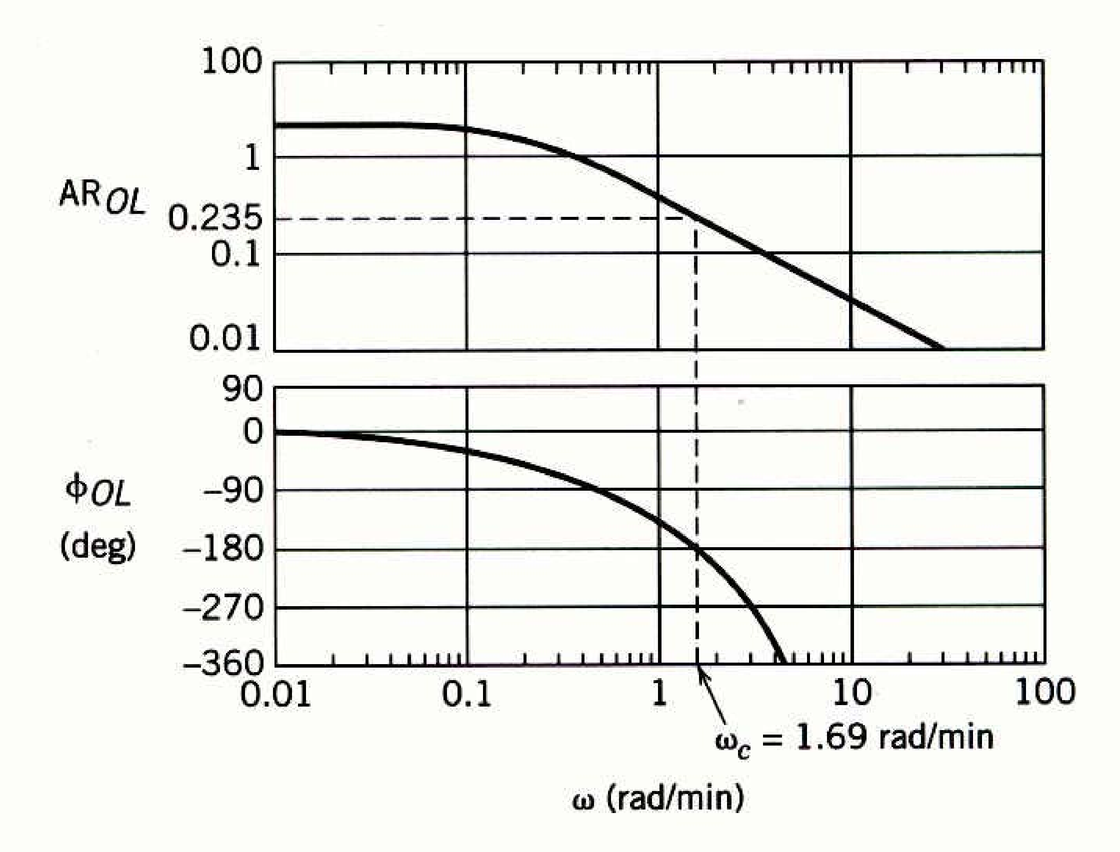
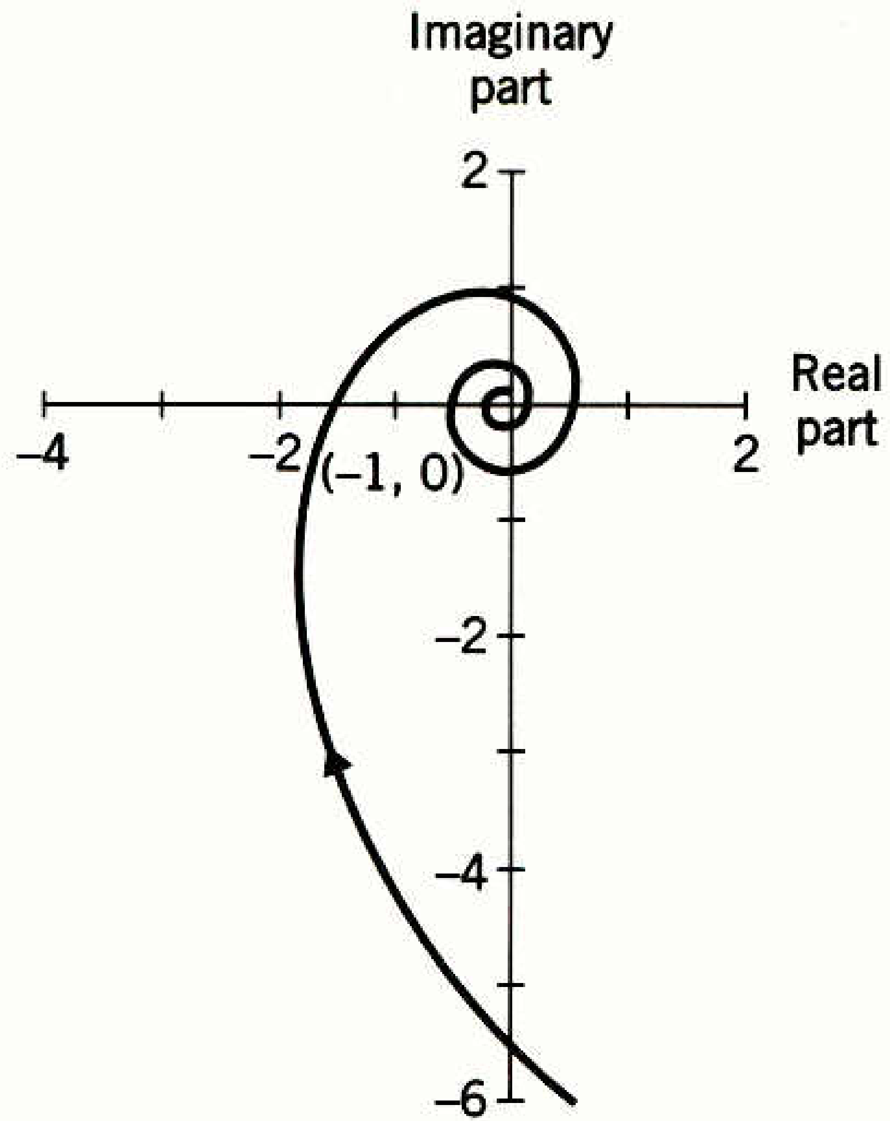


Figure 14.7 Bode plot for Example 14.6, $K_c = 1$.

Figure 14.8 Nyquist plot for Example 14.6, $K_c = 1.5K_{cu} = 6.38$.



Gain and Phase Margins

Let AR_c be the value of the open-loop amplitude ratio at the critical frequency ω_c. Gain margin *GM* is defined as:

$$GM \triangleq \frac{1}{AR_c} \qquad (14\text{-}11)$$

Phase margin *PM* is defined as

$$PM \triangleq 180 + \varphi_g \qquad (14\text{-}12)$$

- The phase margin also provides a measure of relative stability.
- In particular, it indicates how much additional time delay can be included in the feedback loop before instability will occur.
- Denote the additional time delay as $\Delta\theta_{max}$.
- For a time delay of $\Delta\theta_{max}$, the phase angle is $-\Delta\theta_{max}\omega$.

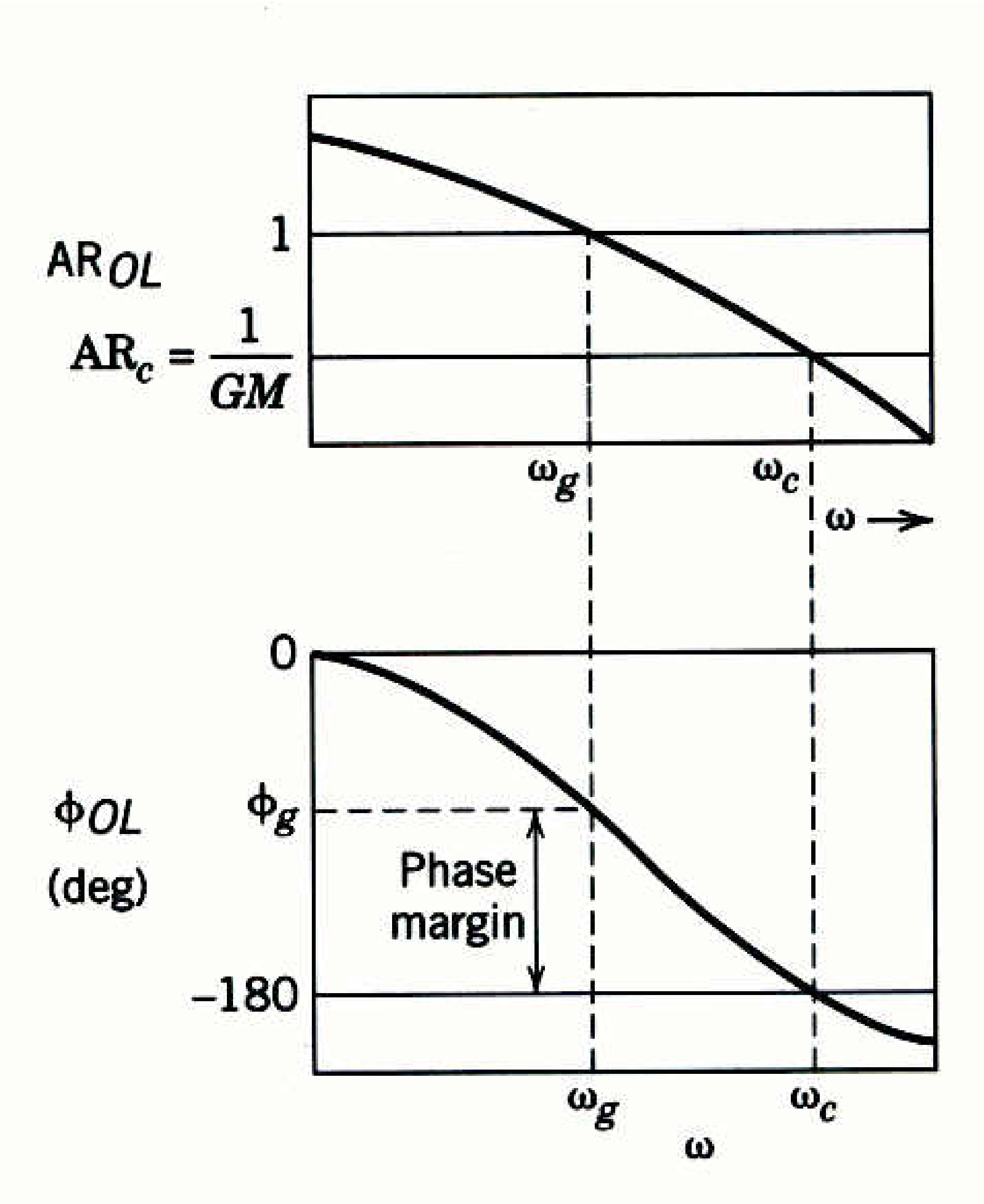


Figure 14.9 Gain and phase margins in Bode plot.

$$PM = \Delta\theta_{max}\,\omega_c\left(\frac{180^\circ}{\pi}\right) \qquad (14\text{-}13)$$

or

$$\Delta\theta_{max} = \left(\frac{\text{PM}}{\omega_c}\right)\left(\frac{\pi}{180^\circ}\right) \qquad (14\text{-}14)$$

where the $\left(\pi/180^\circ\right)$ factor converts *PM* from degrees to radians.

- The specification of phase and gain margins requires a compromise between performance and robustness.
- In general, large values of *GM* and *PM* correspond to sluggish closed-loop responses, while smaller values result in less sluggish, more oscillatory responses.

Guideline. *In general, a well-tuned controller should have a gain margin between 1.7 and 4.0 and a phase margin between 30° and 45°.*

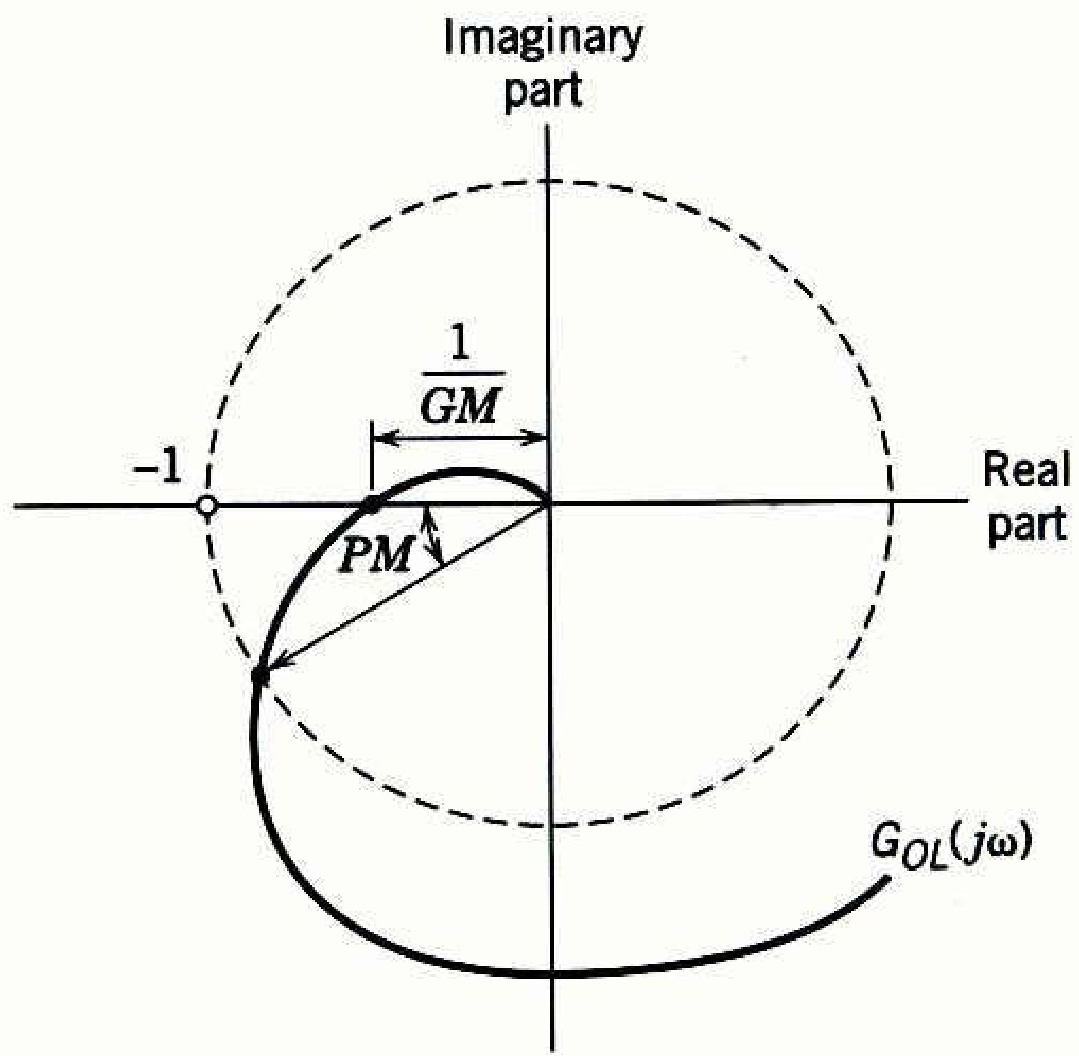


Figure 14.10 Gain and phase margins on a Nyquist plot.

Recognize that these ranges are approximate and that it may not be possible to choose PI or PID controller settings that result in specified GM and PM values.

Example 14.7

For the FOPTD model of Example 14.6, calculate the PID controller settings for the two tuning relations in Table 12.6:

1. Ziegler-Nichols
2. Tyreus-Luyben

Assume that the two PID controllers are implemented in the parallel form with a derivative filter ($\alpha = 0.1$). Plot the open-loop Bode diagram and determine the gain and phase margins for each controller.

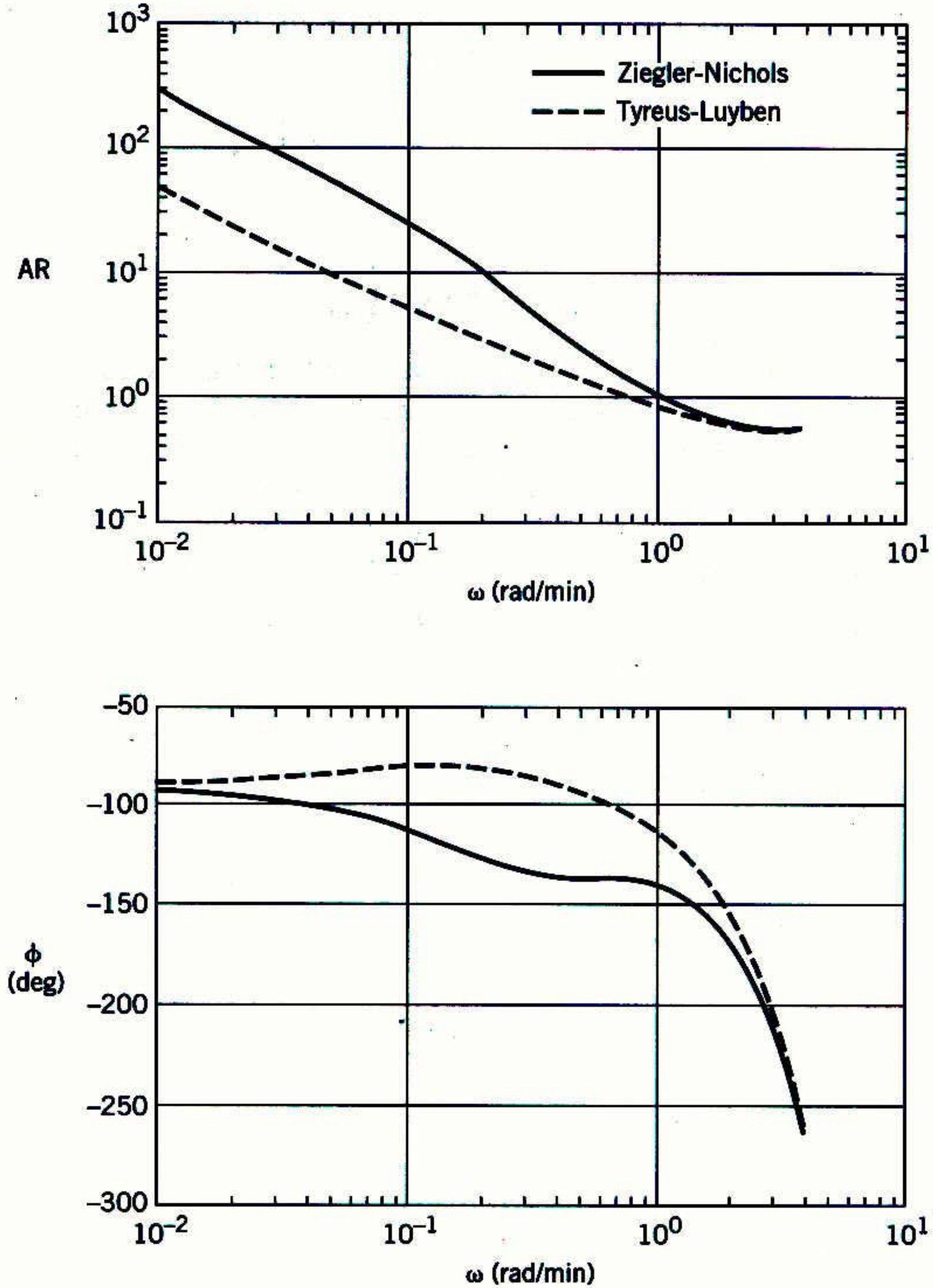


Figure 14.11 Comparison of G_{OL} Bode plots for Example 14.7.

For the Tyreus-Luyben settings, determine the maximum increase in the time delay $\Delta\theta_{max}$ that can occur while still maintaining closed-loop stability.

Solution

From Example 14.6, the ultimate gain is $K_{cu} = 4.25$ and the ultimate period is $P_u = 2\pi/1.69 = 3.72\ \text{min}$. Therefore, the PID controllers have the following settings:

Controller Settings	K_c	τ_I (min)	τ_D (min)
Ziegler-Nichols	2.55	1.86	0.46
Tyreus-Luyben	1.91	8.27	0.59

The open-loop transfer function is:

$$G_{OL} = G_c G_v G_p G_m = G_c \frac{2e^{-s}}{5s+1}$$

Figure 14.11 shows the frequency response of G_{OL} for the two controllers. The gain and phase margins can be determined by inspection of the Bode diagram or by using the MATLAB command, *margin*.

Controller	*GM*	*PM*	ω_c (rad/min)
Ziegler-Nichols	1.6	40°	1.02
Tyreus-Luyben	1.8	76°	0.79

The Tyreus-Luyben controller settings are more conservative owing to the larger gain and phase margins. The value of $\Delta\theta_{max}$ is calculated from Eq. (14-14) and the information in the above table:

$$\Delta\theta_{max} = \frac{(76°)(\pi \text{ rad})}{(0.79 \text{ rad/min})(180°)} = 1.7 \text{ min}$$

Thus, time delay θ can increase by as much as 70% and still maintain closed-loop stability.

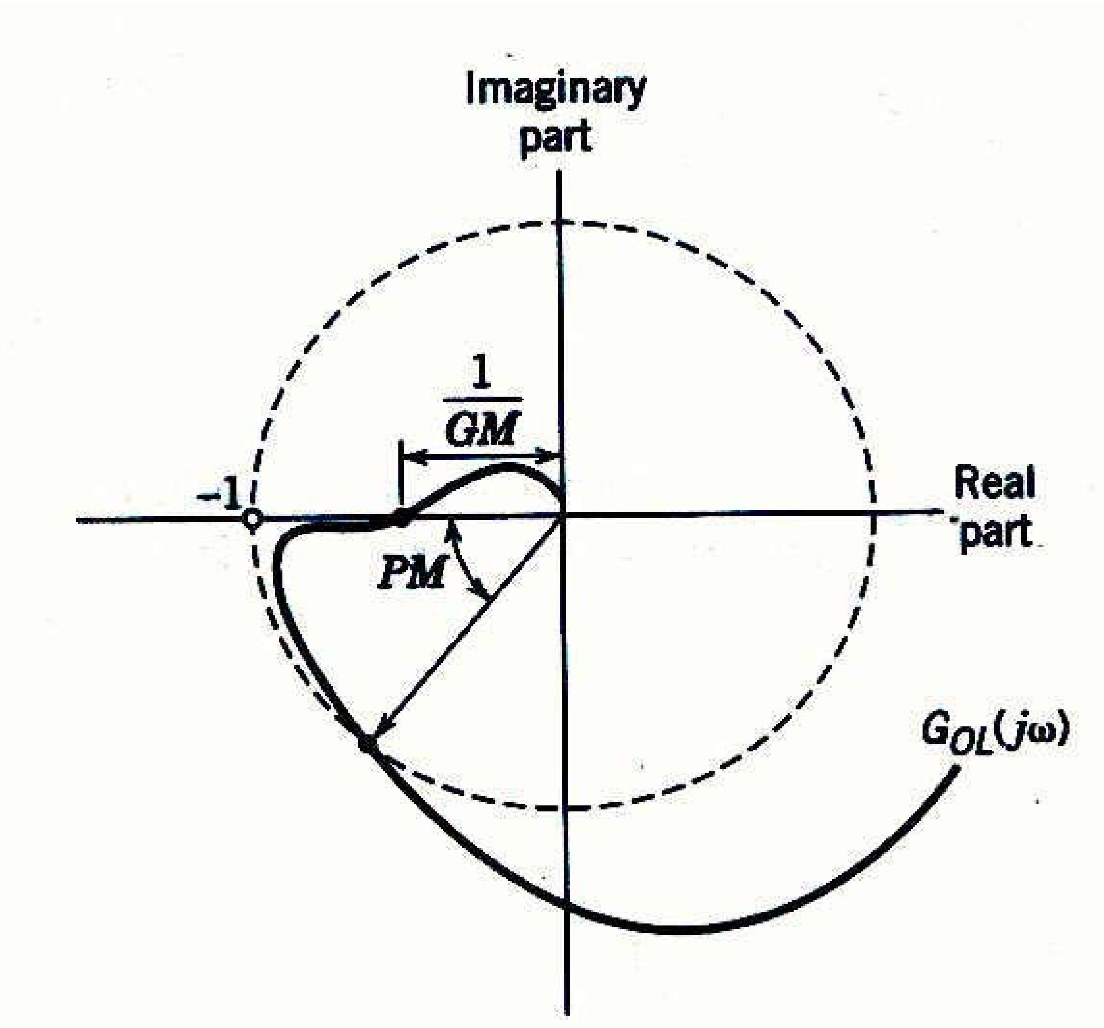


Figure 14.12 Nyquist plot where the gain and phase margins are misleading.

Closed-Loop Frequency Response and Sensitivity Functions

Sensitivity Functions

The following analysis is based on the block diagram in Fig. 14.1. We define G as $G \triangleq G_v G_p G_m$ and assume that $G_m = K_m$ and $G_d = 1$. Two important concepts are now defined:

$$S \triangleq \frac{1}{1+G_c G} \quad \text{sensitivity function} \qquad (14\text{-}15a)$$

$$T \triangleq \frac{G_c G}{1+G_c G} \quad \text{complementary sensitivity function} \qquad (14\text{-}15b)$$

Comparing Fig. 14.1 and Eq. 14-15 indicates that *S* is the closed-loop transfer function for disturbances (Y/D), while *T* is the closed-loop transfer function for set-point changes (Y/Y_{sp}). It is easy to show that:

$$S + T = 1 \tag{14-16}$$

As will be shown in Section 14.6, *S* and *T* provide measures of how sensitive the closed-loop system is to changes in the process.

- Let $|S(j\omega)|$ and $|T(j\omega)|$ denote the amplitude ratios of *S* and *T*, respectively.
- The maximum values of the amplitude ratios provide useful measures of robustness.
- They also serve as control system design criteria, as discussed below.

- Define M_S to be the maximum value of $|S(j\omega)|$ for all frequencies:

$$M_S \triangleq \max_{\omega} |S(j\omega)| \qquad (14\text{-}17)$$

The second robustness measure is M_T, the maximum value of $|T(j\omega)|$:

$$M_T \triangleq \max_{\omega} |T(j\omega)| \qquad (14\text{-}18)$$

M_T is also referred to as the *resonant peak.* Typical amplitude ratio plots for S and T are shown in Fig. 14.13.

It is easy to prove that M_S and M_T are related to the gain and phase margins of Section 14.4 (Morari and Zafiriou, 1989):

$$\text{GM} \geq \frac{M_S}{M_S - 1}, \qquad \text{PM} \geq 2\sin^{-1}\left(\frac{1}{2M_S}\right) \qquad (14\text{-}19)$$

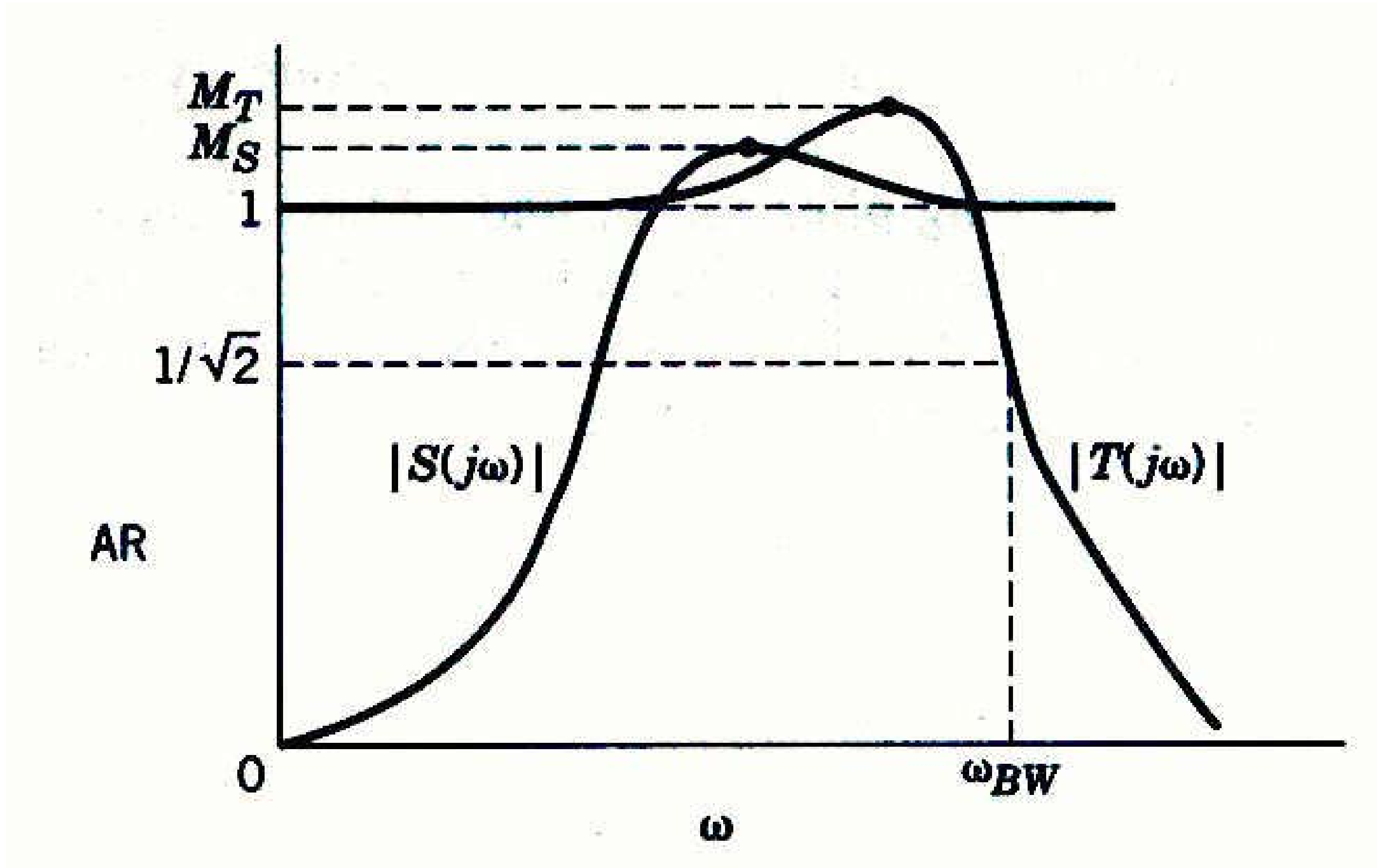


Figure 14.13 Typical *S* and *T* magnitude plots. (Modified from Maciejowski (1998)).

Guideline. *For a satisfactory control system, M_T should be in the range 1.0 – 1.5 and M_S should be in the range of 1.2 – 2.0.*

It is easy to prove that M_S and M_T are related to the gain and phase margins of Section 14.4 (Morari and Zafiriou, 1989):

$$\text{GM} \geq \frac{M_S}{M_S - 1}, \qquad \text{PM} \geq 2\sin^{-1}\left(\frac{1}{2M_S}\right) \qquad (14\text{-}19)$$

$$\text{GM} \geq 1 + \frac{1}{M_T}, \qquad \text{PM} \geq 2\sin^{-1}\left(\frac{1}{2M_T}\right) \qquad (14\text{-}20)$$

Bandwidth

- In this section we introduce an important concept, the *bandwidth*. A typical amplitude ratio plot for T and the corresponding set-point response are shown in Fig. 14.14.
- The definition, the bandwidth ω_{BW} is defined as the frequency at which $|T(j\omega)| = 0.707$.
- The bandwidth indicates the frequency range for which satisfactory set-point tracking occurs. In particular, ω_{BW} is the maximum frequency for a sinusoidal set point to be attenuated by no more than a factor of 0.707.
- The bandwidth is also related to speed of response.
- In general, the bandwidth is (approximately) inversely proportional to the closed-loop settling time.

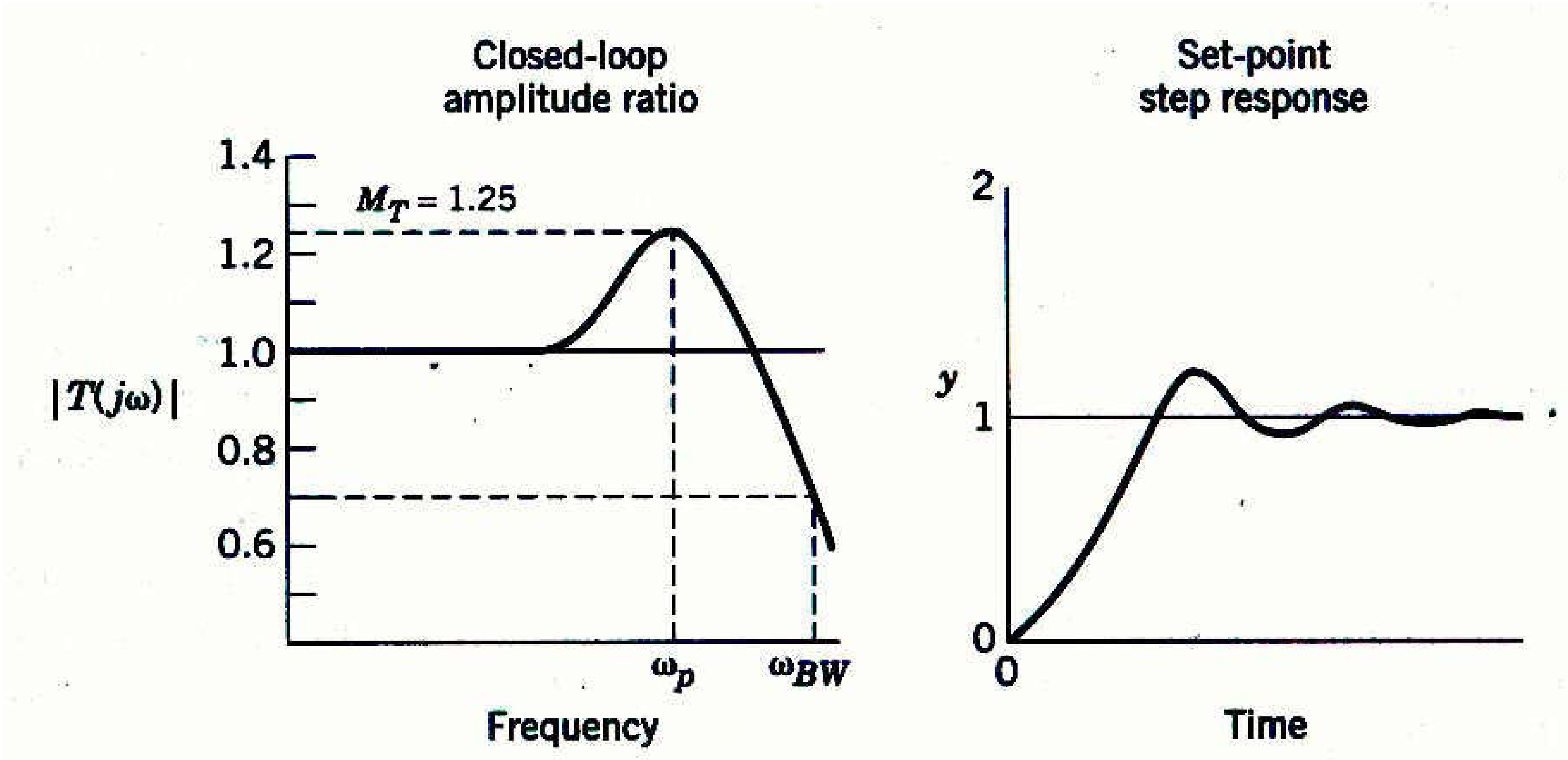


Figure 14.14 Typical closed-loop amplitude ratio $|T(j\omega)|$ and set-point response.

Closed-loop Performance Criteria

Ideally, a feedback controller should satisfy the following criteria.

1. In order to eliminate offset, $|T(j\omega)| \to 1$ as $\omega \to 0$.
2. $|T(j\omega)|$ should be maintained at unity up to as high as frequency as possible. This condition ensures a rapid approach to the new steady state during a set-point change.
3. As indicated in the Guideline, M_T should be selected so that $1.0 < M_T < 1.5$.
4. The bandwidth ω_{BW} and the frequency ω_T at which M_T occurs, should be as large as possible. Large values result in the fast closed-loop responses.

Nichols Chart

The closed-loop frequency response can be calculated analytically from the open-loop frequency response.

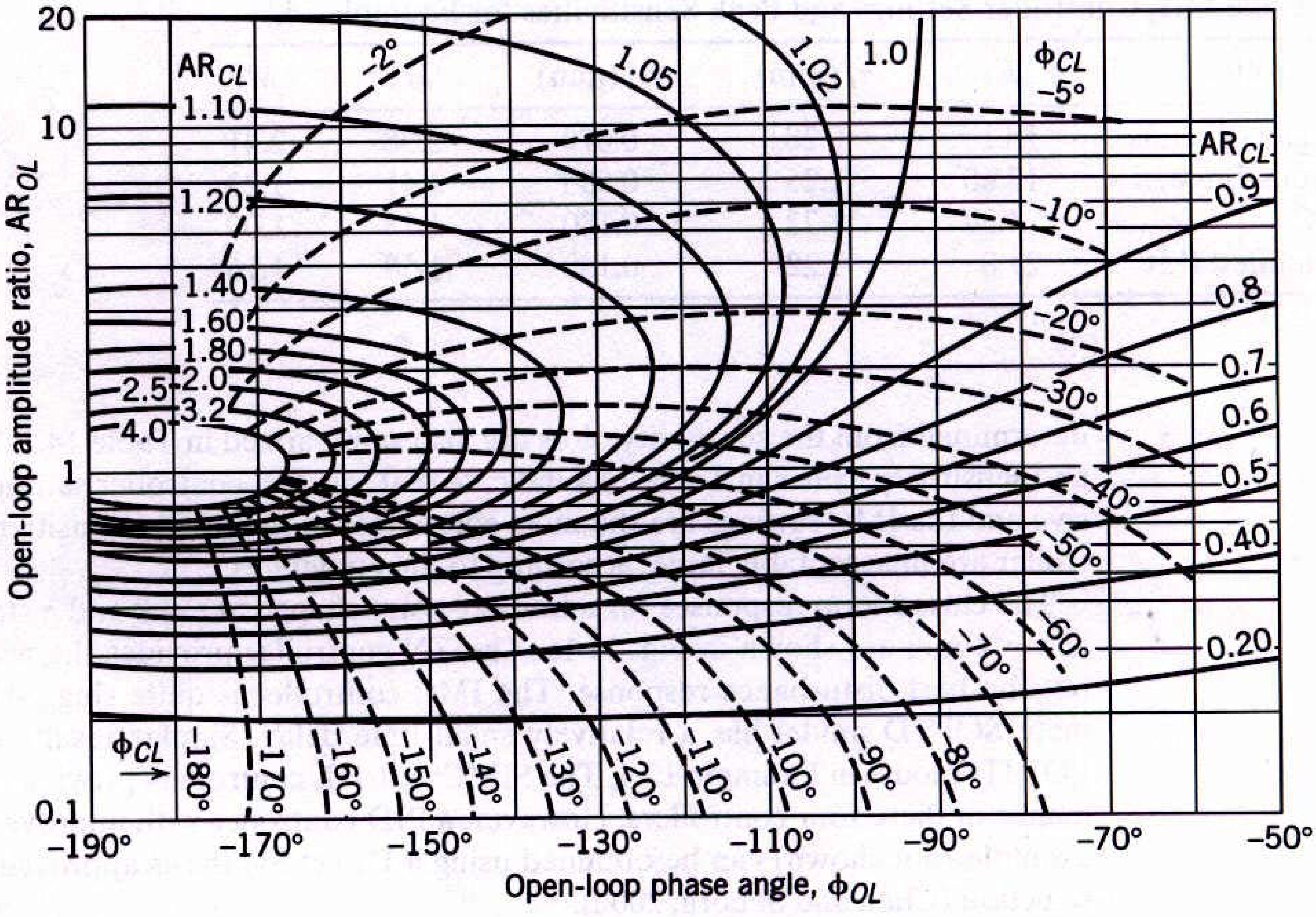


Figure 14.15 A Nichols chart. [The closed-loop amplitude ratio AR_{CL} (———) and phase angle φ_{CL} (– – –) are shown in families of curves.]

Example 14.8

Consider a fourth-order process with a wide range of time constants that have units of minutes (Åström et al., 1998):

$$G = G_v G_p G_m = \frac{1}{(s+1)(0.2s+1)(0.04s+1)(0.008s+1)} \qquad (14\text{-}22)$$

Calculate PID controller settings based on following tuning relations in Chapter 12

a. Ziegler-Nichols tuning (Table 12.6)

b. Tyreus-Luyben tuning (Table 12.6)

c. IMC Tuning with $\tau_c = 0.25$ min (Table 12.1)

d. Simplified IMC (SIMC) tuning (Table 12.5) and a second-order plus time-delay model derived using Skogestad's model approximation method (Section 6.3).

Determine sensitivity peaks M_S and M_T for each controller. Compare the closed-loop responses to step changes in the set-point and the disturbance using the parallel form of the PID controller without a derivative filter:

$$\frac{P'(s)}{E(s)} = K_c\left[1+\frac{1}{\tau_I s}+\tau_D s\right] \qquad (14\text{-}23)$$

Assume that $G_d(s) = G(s)$.

Controller Settings for Example 14.8

Controller	K_c	τ_I (min)	τ_D (min)	M_S	M_T
Ziegler-Nichols	18.1	0.28	0.070	2.38	2.41
Tyreus-Luyben	13.6	1.25	0.089	1.45	1.23
IMC	4.3	1.20	0.167	1.12	1.00
Simplified IMC	21.8	1.22	0.180	1.58	1.16

Chapter 14

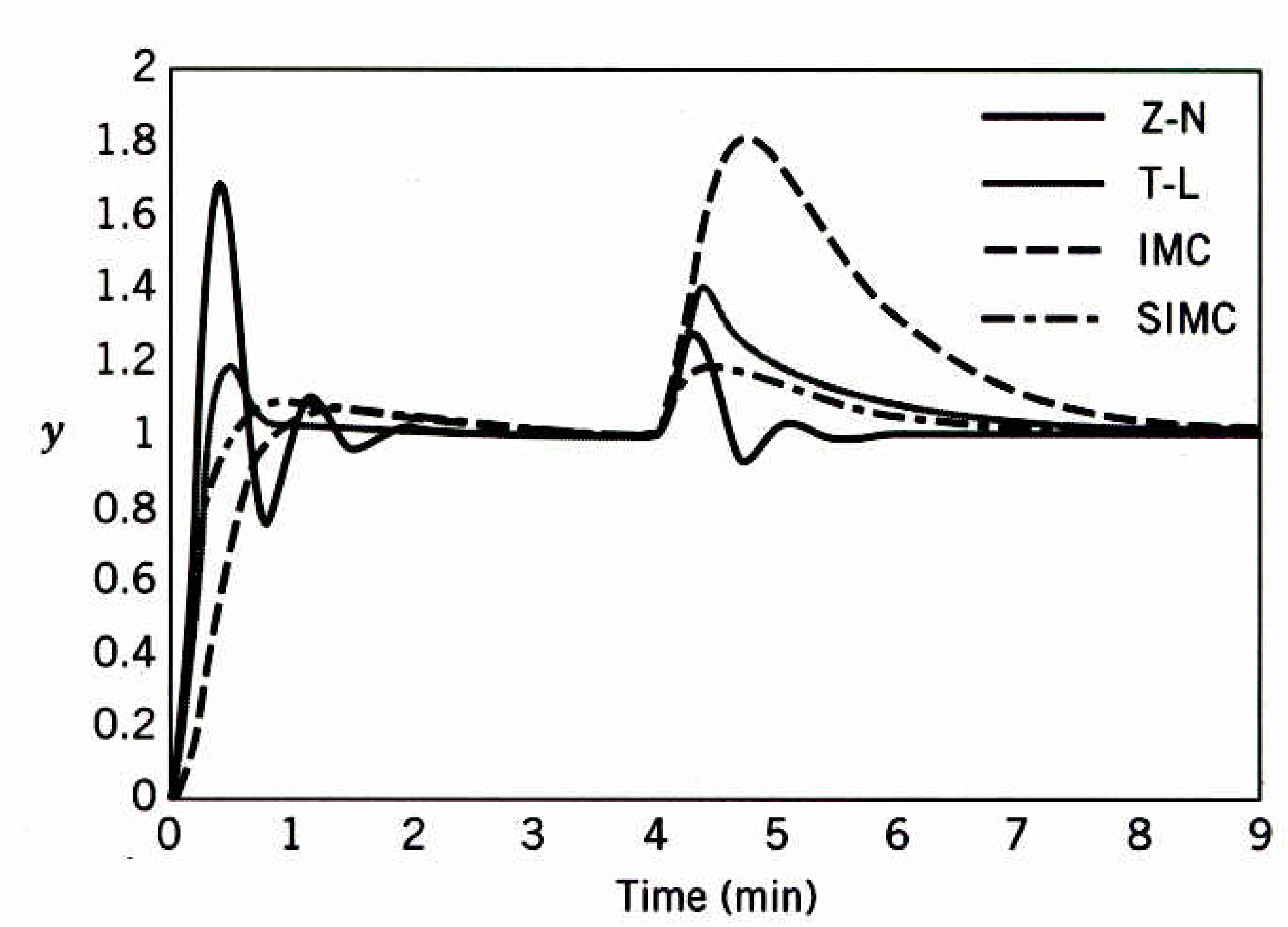


Figure 14.16 Closed-loop responses for Example 14.8. (A set-point change occurs at $t = 0$ and a step disturbance at $t = 4$ min.)

Robustness Analysis

- In order for a control system to function properly, it should not be unduly sensitive to small changes in the process or to inaccuracies in the process model, if a model is used to design the control system.
- A control system that satisfies this requirement is said to be *robust* or *insensitive*.
- It is very important to consider robustness as well as performance in control system design.
- First, we explain why the S and T transfer functions in Eq. 14-15 are referred to as “sensitivity functions”.

Sensitivity Analysis

- In general, the term *sensitivity* refers to the effect that a change in one transfer function (or variable) has on another transfer function (or variable).
- Suppose that G changes from a nominal value G_{p0} to an arbitrary new value, $G_{p0} + dG$.
- This differential change dG causes T to change from its nominal value T_0 to a new value, $T_0 + dT$.
- Thus, we are interested in the ratio of these changes, dT/dG, and also the ratio of the relative changes:

$$\frac{dT / T}{dG / G} \triangleq \text{sensitivity} \qquad (14\text{-}25)$$

We can write the relative sensitivity in an equivalent form:

$$\frac{dT/T}{dG/G} = \left(\frac{dT}{dG}\right)\frac{G}{T} \qquad (14\text{-}26)$$

The derivative in (14-26) can be evaluated after substituting the definition of T in (14-15b):

$$\frac{dT}{dG} = G_c S^2 \qquad (14\text{-}27)$$

Substitute (14-27) into (14-26). Then substituting the definition of S in (14-15a) and rearranging gives the desired result:

$$\frac{dT/T}{dG/G} = \frac{1}{1+G_c G} = S \qquad (14\text{-}28)$$

Chapter 14

- Equation 14-28 indicates that the relative sensitivity is equal to *S*.
- For this reason, *S* is referred to as the *sensitivity function*.
- In view of the important relationship in (14-16), *T* is called the *complementary sensitivity function*.

Effect of Feedback Control on Relative Sensitivity

- Next, we show that feedback reduces sensitivity by comparing the relative sensitivities for open-loop control and closed-loop control.
- By definition, open-loop control occurs when the feedback control loop in Fig. 14.1 is disconnected from the comparator.
- For this condition:

$$\left(\frac{Y}{Y_{sp}}\right)_{OL} = T_{OL} \triangleq G_c G \qquad (14\text{-}29)$$

Chapter 14

Substituting T_{OL} for T in Eq. 14-25 and noting that $dT_{OL}/dG = G_c$ gives:

$$\frac{dT_{OL}/T_{OL}}{dG/G} = \left(\frac{dT_{OL}}{dG}\right)\frac{G}{T_{OL}} = G_c\frac{G}{G_c G} = 1 \qquad (14\text{-}30)$$

- Thus, the relative sensitivity is unity for open-loop control and is equal to S for closed-loop control, as indicated by (14-28).
- Equation 14-15a indicates that $|S| < 1$ if $|G_c G_p| > 1$, which usually occurs over the frequency range of interest.
- Thus, we have identified one of the most important properties of feedback control:
- *Feedback control makes process performance less sensitive to changes in the process.*