Laplace Transforms

- Important analytical method for solving linear ordinary differential equations.
  - Application to nonlinear ODEs? Must linearize first.
- Laplace transforms play a key role in important process control concepts and techniques.
  - Examples:
    - Transfer functions
    - Frequency response
    - Control system design
    - Stability analysis
Definition

The Laplace transform of a function, $f(t)$, is defined as

$$F(s) = \mathcal{L} \left[ f(t) \right] = \int_{0}^{\infty} f(t) e^{-st} \, dt \quad (3-1)$$

where $F(s)$ is the symbol for the Laplace transform, $\mathcal{L}$ is the Laplace transform operator, and $f(t)$ is some function of time, $t$.

*Note*: The $\mathcal{L}$ operator transforms a time domain function $f(t)$ into an $s$ domain function, $F(s)$. $s$ is a complex variable: $s = a + bj$, $j = \sqrt{-1}$
Inverse Laplace Transform, $\mathcal{L}^{-1}$:

By definition, the inverse Laplace transform operator, $\mathcal{L}^{-1}$, converts an $s$-domain function back to the corresponding time domain function:

$$f(t) = \mathcal{L}^{-1}\left[F(s)\right]$$

Important Properties:

Both $\mathcal{L}$ and $\mathcal{L}^{-1}$ are linear operators. Thus,

$$\mathcal{L}\left[a x(t) + b y(t)\right] = a \mathcal{L}\left[x(t)\right] + b \mathcal{L}\left[y(t)\right]$$

$$= a X(s) + b Y(s)$$

(3-3)
where:

- $x(t)$ and $y(t)$ are arbitrary functions
- $a$ and $b$ are constants
- $X(s) \triangleq \mathcal{L}[x(t)]$ and $Y(s) \triangleq \mathcal{L}[y(t)]$

Similarly,

$$\mathcal{L}^{-1}[aX(s) + bY(s)] = ax(t) + by(t)$$
Laplace Transforms of Common Functions

1. Constant Function

Let \( f(t) = a \) (a constant). Then from the definition of the Laplace transform in (3-1),

\[
\mathcal{L}(a) = \int_0^\infty ae^{-st} \, dt = -\frac{a}{s} e^{-st} \bigg|_{0}^{\infty} = 0 - \left( -\frac{a}{s} \right) = \frac{a}{s}
\]  

(3-4)
2. Step Function

The unit step function is widely used in the analysis of process control problems. It is defined as:

\[
S(t) \triangleq \begin{cases} 
0 & \text{for } t < 0 \\
1 & \text{for } t \geq 0 
\end{cases}
\]  

(3-5)

Because the step function is a special case of a “constant”, it follows from (3-4) that

\[
\mathcal{L}[S(t)] = \frac{1}{s}
\]

(3-6)
3. Derivatives

This is a very important transform because derivatives appear in the ODEs we wish to solve. In the text (p.53), it is shown that

\[
\mathcal{L}\left[ \frac{df}{dt} \right] = sF(s) - f(0) \quad \text{(3-9)}
\]

initial condition at \( t = 0 \)

Similarly, for higher order derivatives:

\[
\mathcal{L}\left[ \frac{d^n f}{dt^n} \right] = s^n F(s) - s^{n-1} f(0) - s^{n-2} f^{(1)}(0) - \ldots - sf^{(n-2)}(0) - f^{(n-1)}(0) \quad \text{(3-14)}
\]
where:

- $n$ is an arbitrary positive integer

- $f^{(k)}(0) = \frac{d^k f}{dt^k} \bigg|_{t=0}$

*Special Case: All Initial Conditions are Zero*

Suppose $f(0) = f^{(1)}(0) = \ldots = f^{(n-1)}(0)$. Then

$$\mathcal{L} \left[ \frac{d^n f}{dt^n} \right] = s^n F(s)$$

In process control problems, we usually assume zero initial conditions. *Reason:* This corresponds to the nominal steady state when “deviation variables” are used, as shown in Ch. 4.
4. Exponential Functions

Consider \( f(t) = e^{-bt} \) where \( b > 0 \). Then,

\[
\mathcal{L}\left[ e^{-bt} \right] = \int_0^\infty e^{-bt} e^{-st} dt = \int_0^\infty e^{-(b+s)t} dt
\]

\[
= \frac{1}{b+s} \left[-e^{-(b+s)t}\right]_0^\infty = \frac{1}{s+b} \quad (3-16)
\]

5. Rectangular Pulse Function

It is defined by:

\[
f(t) = \begin{cases} 
0 & \text{for } t < 0 \\
h & \text{for } 0 \leq t < t_w \\
0 & \text{for } t \geq t_w
\end{cases} \quad (3-20)
\]
The Laplace transform of the rectangular pulse is given by

\[ F(s) = \frac{h}{s} \left( 1 - e^{-t_w s} \right) \]  

(3-22)
6. Impulse Function (or Dirac Delta Function)

The impulse function is obtained by taking the limit of the rectangular pulse as its width, $t_w$, goes to zero but holding the area under the pulse constant at one. (i.e., let $h = \frac{1}{t_w}$)

Let, $\delta(t) \triangleq$ impulse function

Then, $\mathcal{L}\left[\delta(t)\right] = 1$

**Solution of ODEs by Laplace Transforms**

**Procedure:**

1. Take the $\mathcal{L}$ of both sides of the ODE.

2. Rearrange the resulting algebraic equation in the $s$ domain to solve for the $\mathcal{L}$ of the output variable, e.g., $Y(s)$.

3. Perform a partial fraction expansion.

4. Use the $\mathcal{L}^{-1}$ to find $y(t)$ from the expression for $Y(s)$. 
Table 3.1. Laplace Transforms

See page 54 of the text.
Example 3.1

Solve the ODE,
\[
5 \frac{dy}{dt} + 4y = 2 \quad y(0) = 1 \quad (3-26)
\]

First, take $\mathcal{L}$ of both sides of (3-26),
\[
5(sY(s) - 1) + 4Y(s) = \frac{2}{s}
\]

Rearrange,
\[
Y(s) = \frac{5s + 2}{s(5s + 4)} \quad (3-34)
\]

Take $\mathcal{L}^{-1}$,
\[
y(t) = \mathcal{L}^{-1} \left[ \frac{5s + 2}{s(5s + 4)} \right]
\]

From Table 3.1,
\[
y(t) = 0.5 + 0.5e^{-0.8t} \quad (3-37)
\]
Partial Fraction Expansions

Basic idea: Expand a complex expression for \( Y(s) \) into simpler terms, each of which appears in the Laplace Transform table. Then you can take the \( \mathcal{L}^{-1} \) of both sides of the equation to obtain \( y(t) \).

Example:

\[
Y(s) = \frac{s + 5}{(s + 1)(s + 4)} \quad (3-41)
\]

Perform a partial fraction expansion (PFE)

\[
\frac{s + 5}{(s + 1)(s + 4)} = \frac{\alpha_1}{s + 1} + \frac{\alpha_2}{s + 4} \quad (3-42)
\]

where coefficients \( \alpha_1 \) and \( \alpha_2 \) have to be determined.
To find $\alpha_1$: Multiply both sides by $s + 1$ and let $s = -1$

$$\therefore \alpha_1 = \frac{s + 5}{s + 4} \bigg|_{s=-1} = \frac{4}{3}$$

To find $\alpha_2$: Multiply both sides by $s + 4$ and let $s = -4$

$$\therefore \alpha_2 = \frac{s + 5}{s + 1} \bigg|_{s=-4} = -\frac{1}{3}$$

A General PFE

Consider a general expression, 

$$Y(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{\pi \left( \sum_{i=1}^{n} (s + b_i) \right)}$$  \hspace{1cm} (3-46a)
Here $D(s)$ is an $n$-th order polynomial with the roots ($s = -b_i$) all being real numbers which are distinct so there are no repeated roots.

The PFE is:

$$Y(s) = \frac{N(s)}{\prod_{i=1}^{n}(s+b_i)} = \sum_{i=1}^{n} \frac{\alpha_i}{s+b_i}$$  \hspace{1cm} (3-46b)

*Note: $D(s)$ is called the “characteristic polynomial”.

**Special Situations:**

Two other types of situations commonly occur when $D(s)$ has:

i) Complex roots: e.g., $b_i = 3 \pm 4j$  \hspace{1cm} ($j \triangleq \sqrt{-1}$)

ii) Repeated roots (e.g., $b_1 = b_2 = -3$)

For these situations, the PFE has a different form. See SEM text (pp. 61-64) for details.
Example 3.2 (continued)

Recall that the ODE, \( \ddot{y} + 6\dot{y} + 11y + 6y = 1 \) with zero initial conditions resulted in the expression

\[
Y(s) = \frac{1}{s(s^3 + 6s^2 + 11s + 6)} \quad (3-40)
\]

The denominator can be factored as

\[
s(s^3 + 6s^2 + 11s + 6) = s(s + 1)(s + 2)(s + 3) \quad (3-50)
\]

*Note*: Normally, numerical techniques are required in order to calculate the roots.

The PFE for (3-40) is

\[
Y(s) = \frac{1}{s(s + 1)(s + 2)(s + 3)} = \frac{\alpha_1}{s} + \frac{\alpha_2}{s + 1} + \frac{\alpha_3}{s + 2} + \frac{\alpha_4}{s + 3} \quad (3-51)
\]
Solve for coefficients to get

\[ \alpha_1 = \frac{1}{6}, \quad \alpha_2 = -\frac{1}{2}, \quad \alpha_3 = \frac{1}{2}, \quad \alpha_4 = -\frac{1}{6} \]

(For example, find \( \alpha \), by multiplying both sides by \( s \) and then setting \( s = 0 \).)

Substitute numerical values into (3-51):

\[ Y(s) = \frac{1/6}{s} - \frac{1/2}{s + 1} + \frac{1/2}{s + 2} - \frac{1/6}{s + 3} \]

Take \( \mathcal{L}^{-1} \) of both sides:

\[ \mathcal{L}^{-1}\left[ Y(s) \right] = \mathcal{L}^{-1}\left[ \frac{1/6}{s} \right] - \mathcal{L}^{-1}\left[ \frac{1/2}{s + 1} \right] + \mathcal{L}^{-1}\left[ \frac{1/2}{s + 2} \right] - \mathcal{L}^{-1}\left[ \frac{1/6}{s + 3} \right] \]

From Table 3.1,

\[ y(t) = \frac{1}{6} - \frac{1}{2}e^{-t} + \frac{1}{2}e^{-2t} - \frac{1}{6}e^{-3t} \quad (3-52) \]
Important Properties of Laplace Transforms

1. Final Value Theorem

It can be used to find the steady-state value of a closed loop system (providing that a steady-state value exists.

Statement of FVT:

$$\lim_{t \to \infty} y(t) = \lim_{s \to 0} sY(s)$$

providing that the limit exists (is finite) for all \( \text{Re}(s) \geq 0 \), where \( \text{Re}(s) \) denotes the real part of complex variable, \( s \).
Example:

Suppose,

\[ Y(s) = \frac{5s + 2}{s(5s + 4)} \]  \hspace{1cm} (3-34)

Then,

\[ y(\infty) = \lim_{t \to \infty} y(t) = \lim_{s \to 0} \left[ \frac{5s + 2}{5s + 4} \right] = 0.5 \]

2. Time Delay

Time delays occur due to fluid flow, time required to do an analysis (e.g., gas chromatograph). The delayed signal can be represented as

\[ y(t - \theta) \quad \theta = \text{time delay} \]

Also,

\[ \mathcal{L}\left[ y(t - \theta) \right] = e^{-\theta s}Y(s) \]