

Frequency Response Analysis

Sinusoidal Forcing of a First-Order Process

For a first-order transfer function with gain K and time constant τ , the response to a general sinusoidal input, $x(t) = A \sin \omega t$ is:

$$y(t) = \frac{KA}{\omega^2 \tau^2 + 1} \left(\omega \tau e^{-t/\tau} - \omega \tau \cos \omega t + \sin \omega t \right) \quad (5-25)$$

Note that $y(t)$ and $x(t)$ are in deviation form. The *long-time response*, $y_l(t)$, can be written as:

$$y_l(t) = \frac{KA}{\sqrt{\omega^2 \tau^2 + 1}} \sin(\omega t + \varphi) \text{ for } t \rightarrow \infty \quad (13-1)$$

where:

$$\varphi = -\tan^{-1}(\omega \tau)$$

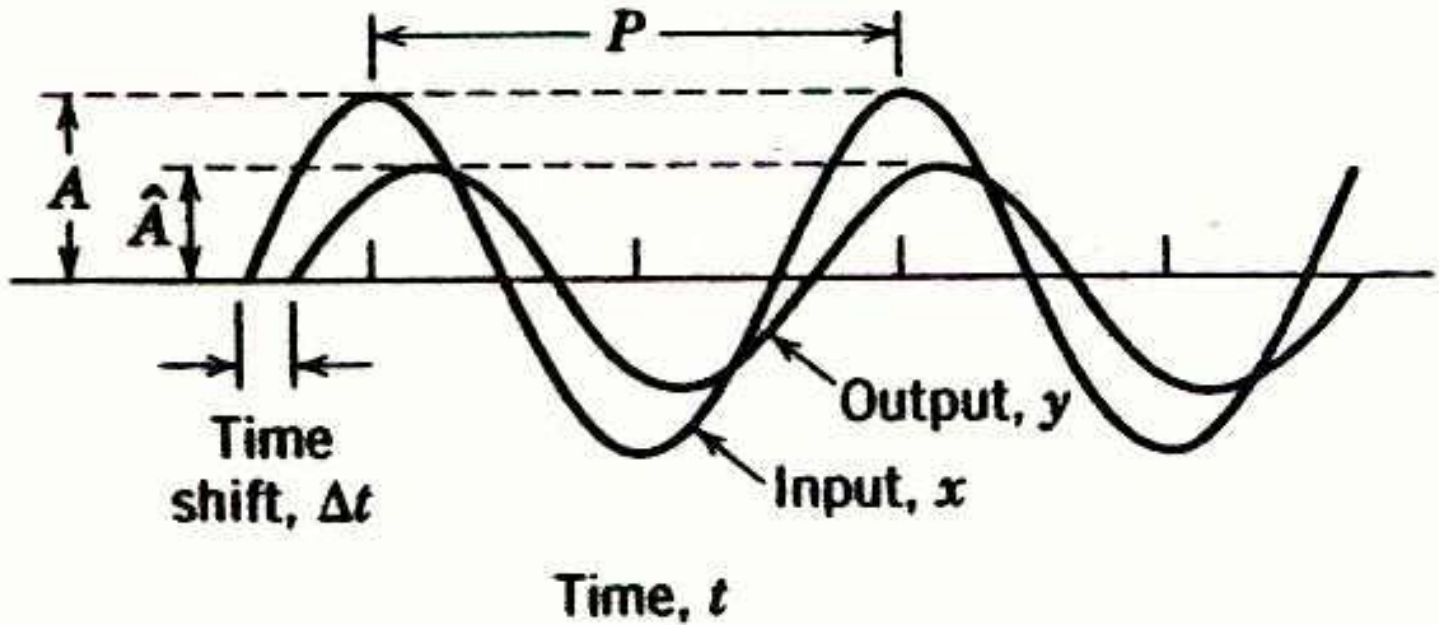


Figure 13.1 Attenuation and time shift between input and output sine waves ($K=1$). The phase angle ϕ of the output signal is given by $\phi = -\text{Time shift} / P \times 360^\circ$, where Δt is the (period) shift and P is the period of oscillation.

Frequency Response Characteristics of a First-Order Process

For $x(t) = A \sin \omega t$, $y_\ell(t) = \hat{A} \sin(\omega t + \varphi)$ as $t \rightarrow \infty$ where :

$$\hat{A} = \frac{KA}{\sqrt{\omega^2 \tau^2 + 1}} \quad \text{and} \quad \varphi = -\tan^{-1}(\omega \tau)$$

1. The output signal is a sinusoid that has the same frequency, ω , as the input signal, $x(t) = A \sin \omega t$.
2. The amplitude of the output signal, \hat{A} , is a function of the frequency ω and the input amplitude, A :

$$\hat{A} = \frac{KA}{\sqrt{\omega^2 \tau^2 + 1}} \quad (13-2)$$

3. The output has a phase shift, φ , relative to the input. The amount of phase shift depends on ω .

Dividing both sides of (13-2) by the input signal amplitude A yields the *amplitude ratio* (AR)

$$\text{AR} = \frac{\hat{A}}{A} = \frac{K}{\sqrt{\omega^2 \tau^2 + 1}} \quad (13-3a)$$

which can, in turn, be divided by the process gain to yield the *normalized amplitude ratio* (AR_N)

$$\text{AR}_N = \frac{1}{\sqrt{\omega^2 \tau^2 + 1}} \quad (13-3b)$$

Shortcut Method for Finding the Frequency Response

The shortcut method consists of the following steps:

Step 1. Set $s=j\omega$ in $G(s)$ to obtain $G(j\omega)$.

Step 2. Rationalize $G(j\omega)$; We want to express it in the form.

$$G(j\omega)=R + jI$$

where R and I are functions of ω . Simplify $G(j\omega)$ by multiplying the numerator and denominator by the complex conjugate of the denominator.

Step 3. The amplitude ratio and phase angle of $G(s)$ are given by:

Memorize \Rightarrow

$$\begin{aligned} \text{AR} &= \sqrt{R^2 + I^2} \\ \varphi &= \tan^{-1}(R / I) \end{aligned}$$

Example 13.1

Find the frequency response of a first-order system, with

$$G(s) = \frac{1}{\tau s + 1} \quad (13-16)$$

Solution

First, substitute $s = j\omega$ in the transfer function

$$G(j\omega) = \frac{1}{\tau j\omega + 1} = \frac{1}{j\omega\tau + 1} \quad (13-17)$$

Then multiply both numerator and denominator by the complex conjugate of the denominator, that is, $-j\omega\tau + 1$

$$\begin{aligned} G(j\omega) &= \frac{-j\omega\tau + 1}{(j\omega\tau + 1)(-j\omega\tau + 1)} = \frac{-j\omega\tau + 1}{\omega^2\tau^2 + 1} \\ &= \frac{1}{\omega^2\tau^2 + 1} + j \frac{(-\omega\tau)}{\omega^2\tau^2 + 1} = R + jI \end{aligned} \quad (13-18)$$

where:
$$R = \frac{1}{\omega^2 \tau^2 + 1} \quad (13-19a)$$

$$I = \frac{-\omega \tau}{\omega^2 \tau^2 + 1} \quad (13-19b)$$

From Step 3 of the Shortcut Method,

$$AR = \sqrt{R^2 + I^2} = \sqrt{\left(\frac{1}{\omega^2 \tau^2 + 1}\right)^2 + \left(\frac{-\omega \tau}{\omega^2 \tau^2 + 1}\right)^2}$$

or

$$AR = \sqrt{\frac{(1 + \omega^2 \tau^2)}{(\omega^2 \tau^2 + 1)^2}} = \frac{1}{\sqrt{\omega^2 \tau^2 + 1}} \quad (13-20a)$$

Also,

$$\phi = \tan^{-1} \left(\frac{I}{R} \right) = \tan^{-1} (-\omega \tau) = -\tan^{-1} (\omega \tau) \quad (13-20b)$$

Complex Transfer Functions

Consider a complex transfer $G(s)$,

$$G(s) = \frac{G_a(s) G_b(s) G_c(s) \cdots}{G_1(s) G_2(s) G_3(s) \cdots} \quad (13-22)$$

Substitute $s=j\omega$,

$$G(j\omega) = \frac{G_a(j\omega) G_b(j\omega) G_c(j\omega) \cdots}{G_1(j\omega) G_2(j\omega) G_3(j\omega) \cdots} \quad (13-23)$$

From complex variable theory, we can express the magnitude and angle of $G(j\omega)$ as follows:

$$|G(j\omega)| = \frac{|G_a(j\omega)| |G_b(j\omega)| |G_c(j\omega)| \cdots}{|G_1(j\omega)| |G_2(j\omega)| |G_3(j\omega)| \cdots} \quad (13-24a)$$

$$\begin{aligned} \angle G(j\omega) = & \angle G_a(j\omega) + \angle G_b(j\omega) + \angle G_c(j\omega) + \cdots \\ & - [\angle G_1(j\omega) + \angle G_2(j\omega) + \angle G_3(j\omega) + \cdots] \end{aligned} \quad (13-24b)$$

Bode Diagrams

- A special graph, called the *Bode diagram* or *Bode plot*, provides a convenient display of the frequency response characteristics of a transfer function model. It consists of plots of AR and φ as a function of ω .
- Ordinarily, ω is expressed in units of radians/time.

Bode Plot of A First-order System

Recall:

$$AR_N = \frac{1}{\sqrt{\omega^2 \tau^2 + 1}} \quad \text{and} \quad \varphi = -\tan^{-1}(\omega\tau)$$

- **At low frequencies** ($\omega \rightarrow 0$ and $\omega\tau \ll 1$):

$$AR_N = 1 \quad \text{and} \quad \varphi = 0$$

- **At high frequencies** ($\omega \rightarrow \infty$ and $\omega\tau \gg 1$):

$$AR_N = 1/\omega\tau \quad \text{and} \quad \varphi = -90^\circ$$

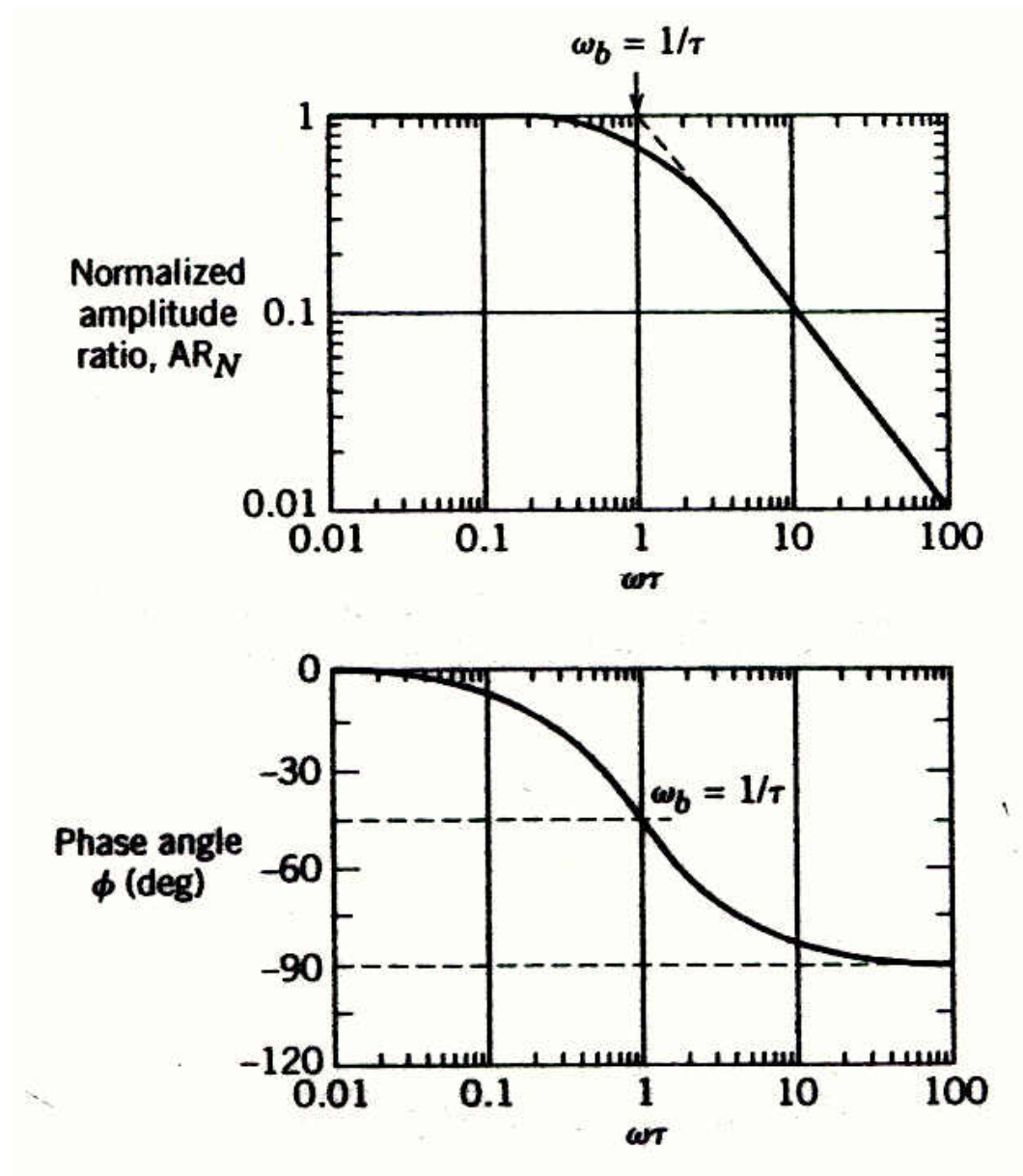


Figure 13.2 Bode diagram for a first-order process.

- Note that the asymptotes intersect at $\omega = \omega_b = 1/\tau$, known as the *break frequency* or *corner frequency*. Here the value of AR_N from (13-21) is:

$$AR_N(\omega = \omega_b) = \frac{1}{\sqrt{1+1}} = 0.707 \quad (13-30)$$

- Some books and software defined AR differently, in terms of *decibels*. The amplitude ratio in decibels AR_d is defined as

$$AR_d = 20 \log AR \quad (13-33)$$

Integrating Elements

The transfer function for an integrating element was given in Chapter 5:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{K}{s} \quad (5-34)$$

$$\text{AR} = |G(j\omega)| = \left| \frac{K}{j\omega} \right| = \frac{K}{\omega} \quad (13-34)$$

$$\varphi = \angle G(j\omega) = \angle K - \angle(\infty) = -90^\circ \quad (13-35)$$

Second-Order Process

A general transfer function that describes any underdamped, critically damped, or overdamped second-order system is

$$G(s) = \frac{K}{\tau^2 s^2 + 2\zeta\tau s + 1} \quad (13-40)$$

Substituting $s = j\omega$ and rearranging yields:

$$AR = \frac{K}{\sqrt{(1 - \omega^2 \tau^2)^2 + (2\omega \tau \zeta)^2}} \quad (13-41a)$$

$$\phi = \tan^{-1} \left[\frac{-2\omega \tau \zeta}{1 - \omega^2 \tau^2} \right] \quad (13-41b)$$

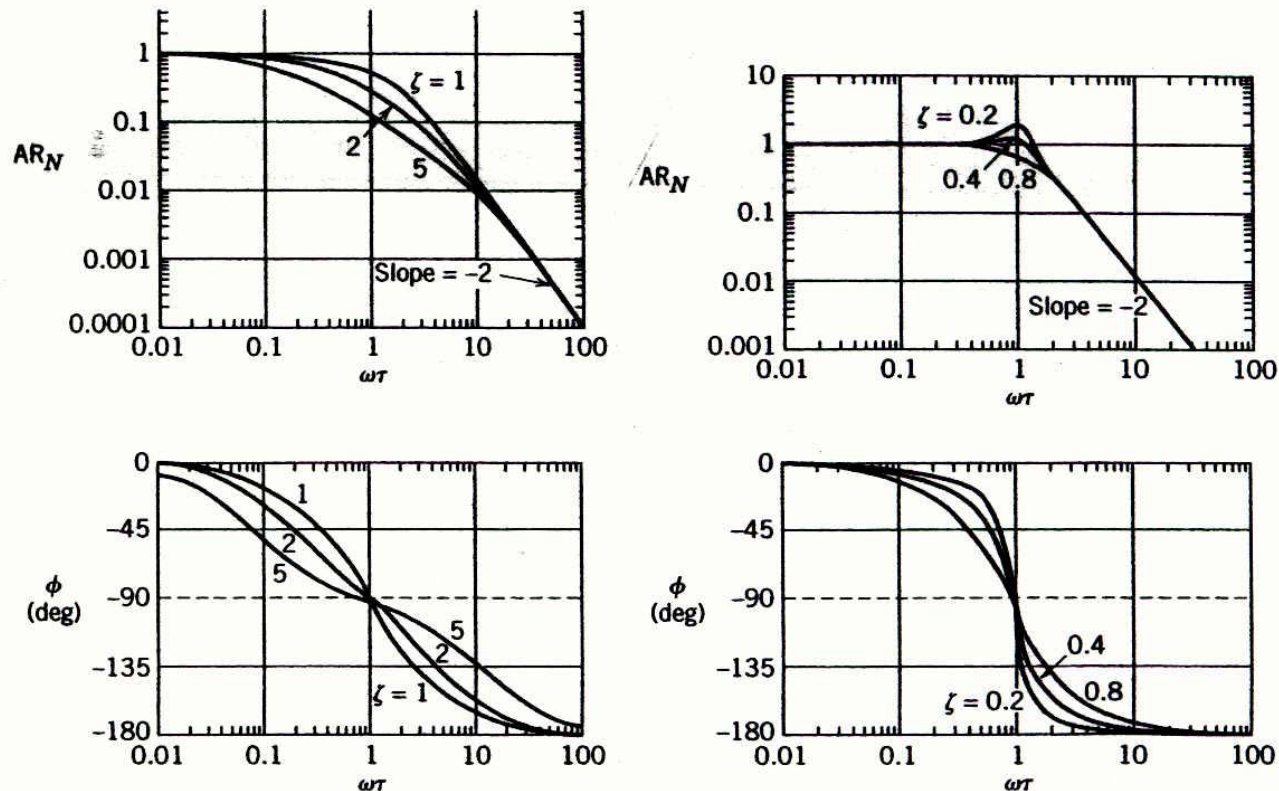


Figure 13.3 Bode diagrams for second-order processes.

Time Delay

Its frequency response characteristics can be obtained by substituting $s = j\omega$,

$$G(j\omega) = e^{-j\omega\theta} \quad (13-53)$$

which can be written in rational form by substitution of the Euler identity,

$$G(j\omega) = e^{-j\omega\theta} = \cos \omega\theta - j \sin \omega\theta \quad (13-54)$$

From (13-54)

$$\text{AR} = |G(j\omega)| = \sqrt{\cos^2 \omega\theta + \sin^2 \omega\theta} = 1 \quad (13-55)$$

$$\phi = \angle G(j\omega) = \tan^{-1} \left(-\frac{\sin \omega\theta}{\cos \omega\theta} \right)$$

or

$$\phi = -\omega\theta \quad (13-56)$$

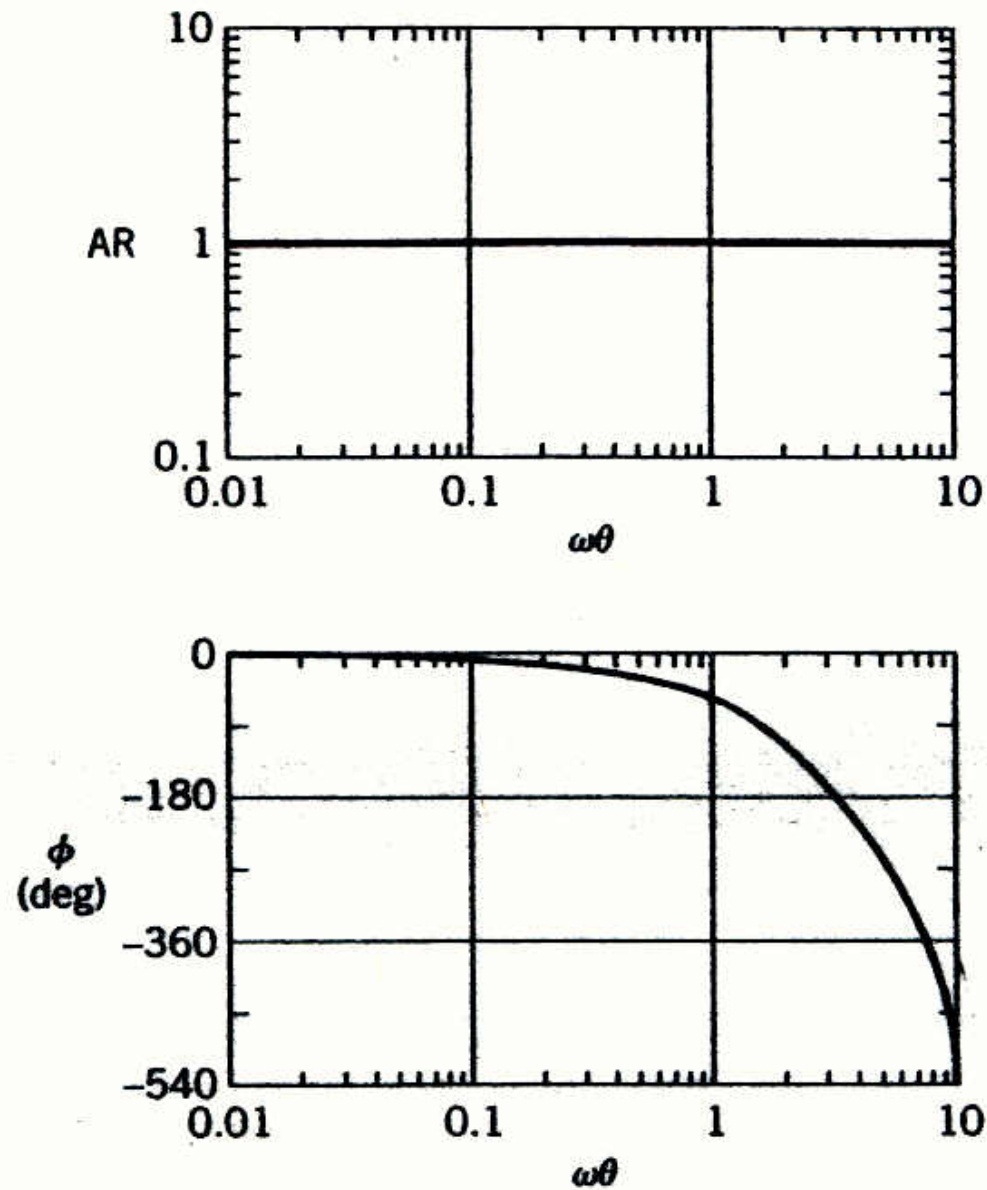


Figure 13.6 Bode diagram for a time delay, $e^{-\theta s}$.

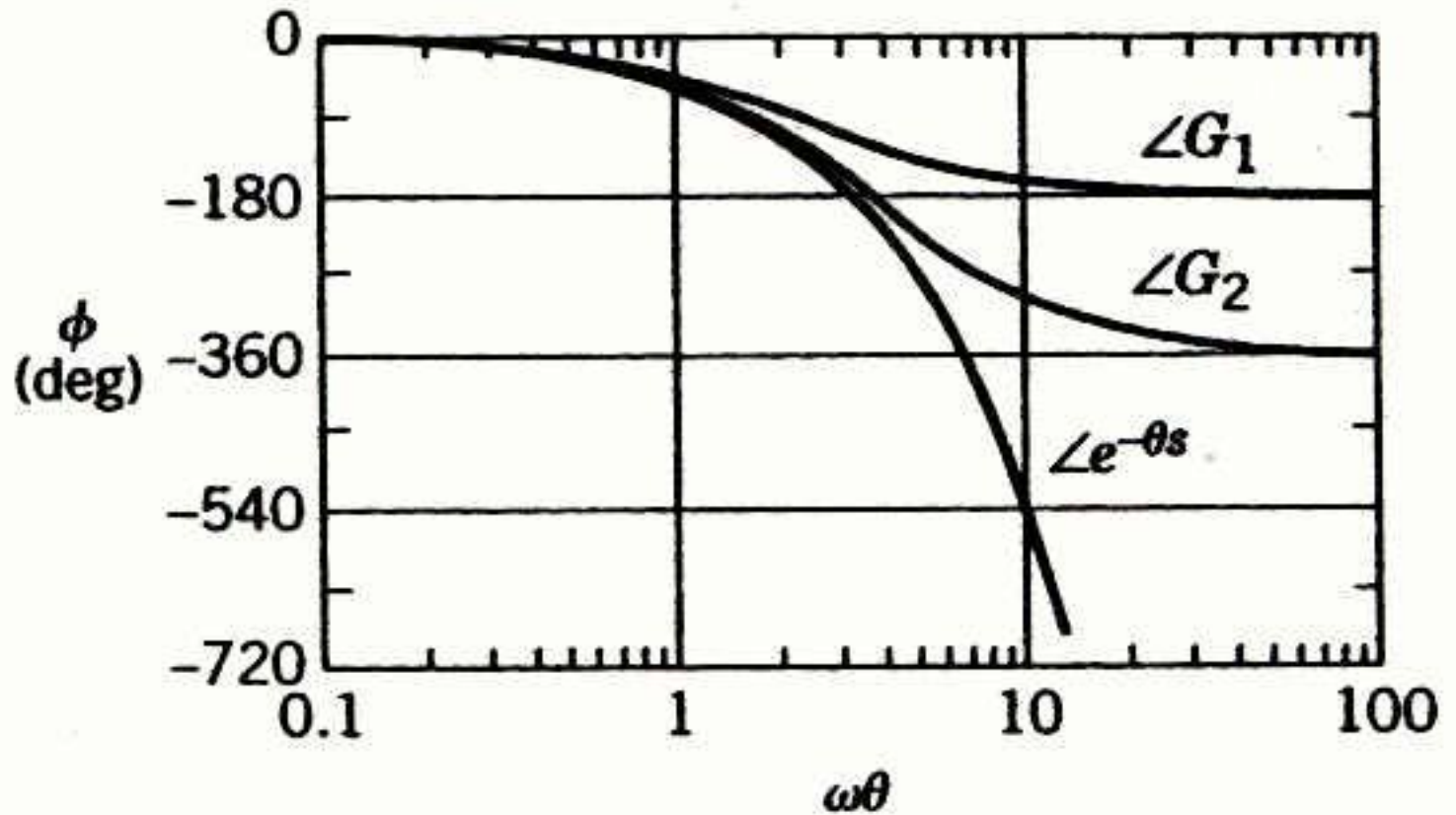


Figure 13.7 Phase angle plots for $e^{-\theta s}$ and for the 1/1 and 2/2 Padé approximations (G_1 is 1/1; G_2 is 2/2).

Process Zeros

Consider a process zero term,

$$G(s) = K(s\tau + 1)$$

Substituting $s=j\omega$ gives

$$G(j\omega) = K(j\omega\tau + 1)$$

Thus:

$$\text{AR} = |G(j\omega)| = K\sqrt{\omega^2\tau^2 + 1}$$

$$\phi = \angle G(j\omega) = +\tan^{-1}(\omega\tau)$$

Note: In general, a multiplicative constant (e.g., K) changes the AR by a factor of K without affecting ϕ .

Frequency Response Characteristics of Feedback Controllers

Proportional Controller. Consider a proportional controller with positive gain

$$G_c(s) = K_c \quad (13-57)$$

In this case $|G_c(j\omega)| = K_c$, which is independent of ω .
Therefore,

$$AR_c = K_c \quad (13-58)$$

and

$$\varphi_c = 0^\circ \quad (13-59)$$

Proportional-Integral Controller. A proportional-integral (PI) controller has the transfer function (cf. Eq. 8-9),

$$G_c(s) = K_c \left(1 + \frac{1}{\tau_I s} \right) = K_c \left(\frac{\tau_I s + 1}{\tau_I s} \right) \quad (13-60)$$

Substitute $s=j\omega$:

$$G_c(j\omega) = K_c \left(1 + \frac{1}{\tau_I j\omega} \right) = K_c \left(\frac{j\omega\tau_I + 1}{j\omega\tau_I} \right) = K_c \left(1 - \frac{1}{\tau_I \omega} j \right)$$

Thus, the amplitude ratio and phase angle are:

$$\text{AR}_c = |G_c(j\omega)| = K_c \sqrt{1 + \frac{1}{(\omega\tau_I)^2}} = K_c \frac{\sqrt{(\omega\tau_I)^2 + 1}}{\omega\tau_I} \quad (13-62)$$

$$\phi_c = \angle G_c(j\omega) = \tan^{-1}(-1/\omega\tau_I) = \tan^{-1}(\omega\tau_I) - 90^\circ \quad (13-63)$$

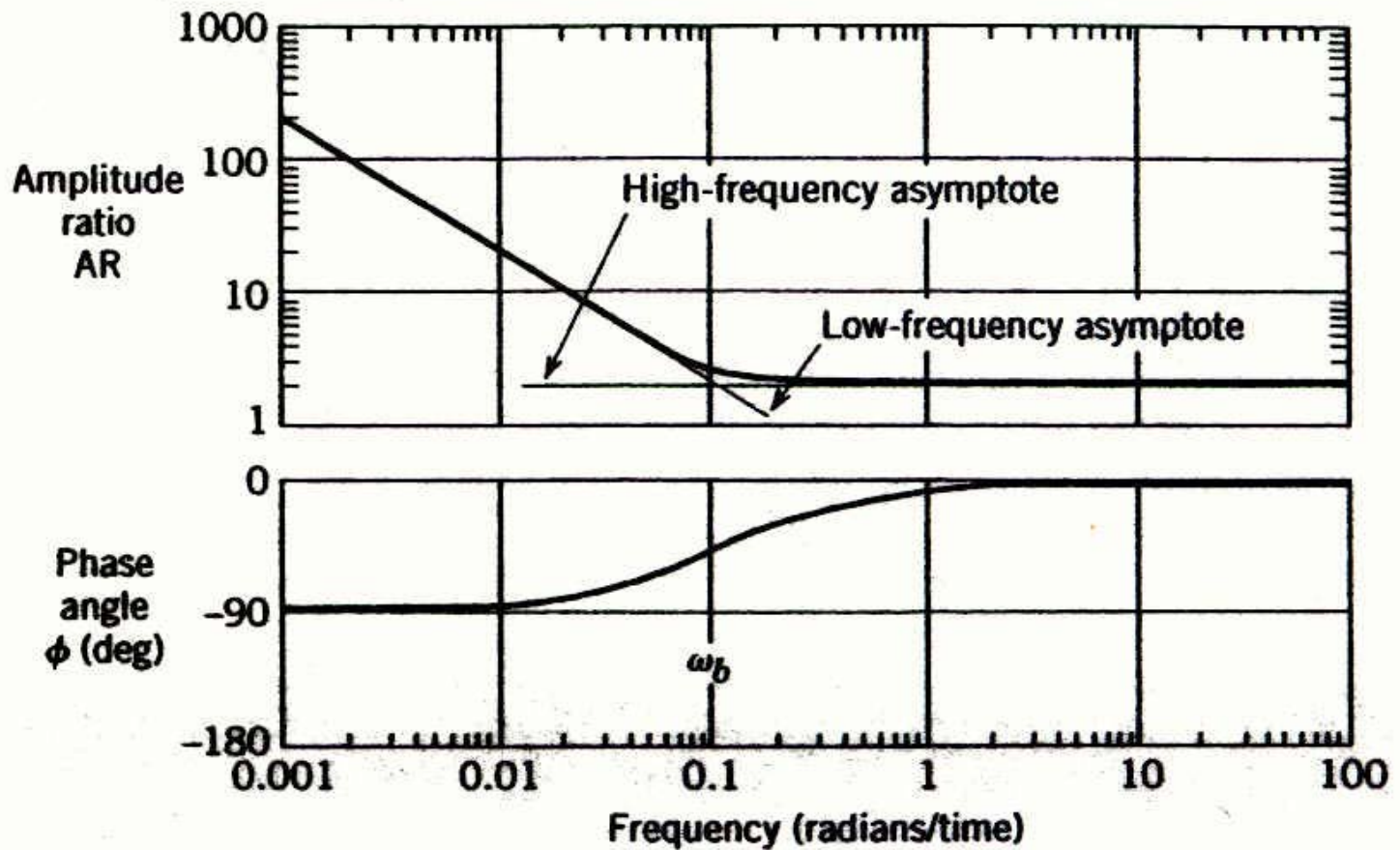


Figure 13.9 Bode plot of a PI controller, $G_c(s) = 2 \left(\frac{10s + 1}{10s} \right)$

Ideal Proportional-Derivative Controller. For the ideal proportional-derivative (PD) controller (cf. Eq. 8-11)

$$G_c(s) = K_c(1 + \tau_D s) \quad (13-64)$$

The frequency response characteristics are similar to those of a LHP zero:

$$\text{AR}_c = K_c \sqrt{(\omega \tau_D)^2 + 1} \quad (13-65)$$

$$\phi = \tan^{-1}(\omega \tau_D) \quad (13-66)$$

Proportional-Derivative Controller with Filter. The PD controller is most often realized by the transfer function

$$G_c(s) = K_c \left(\frac{\tau_D s + 1}{\alpha \tau_D s + 1} \right) \quad (13-67)$$

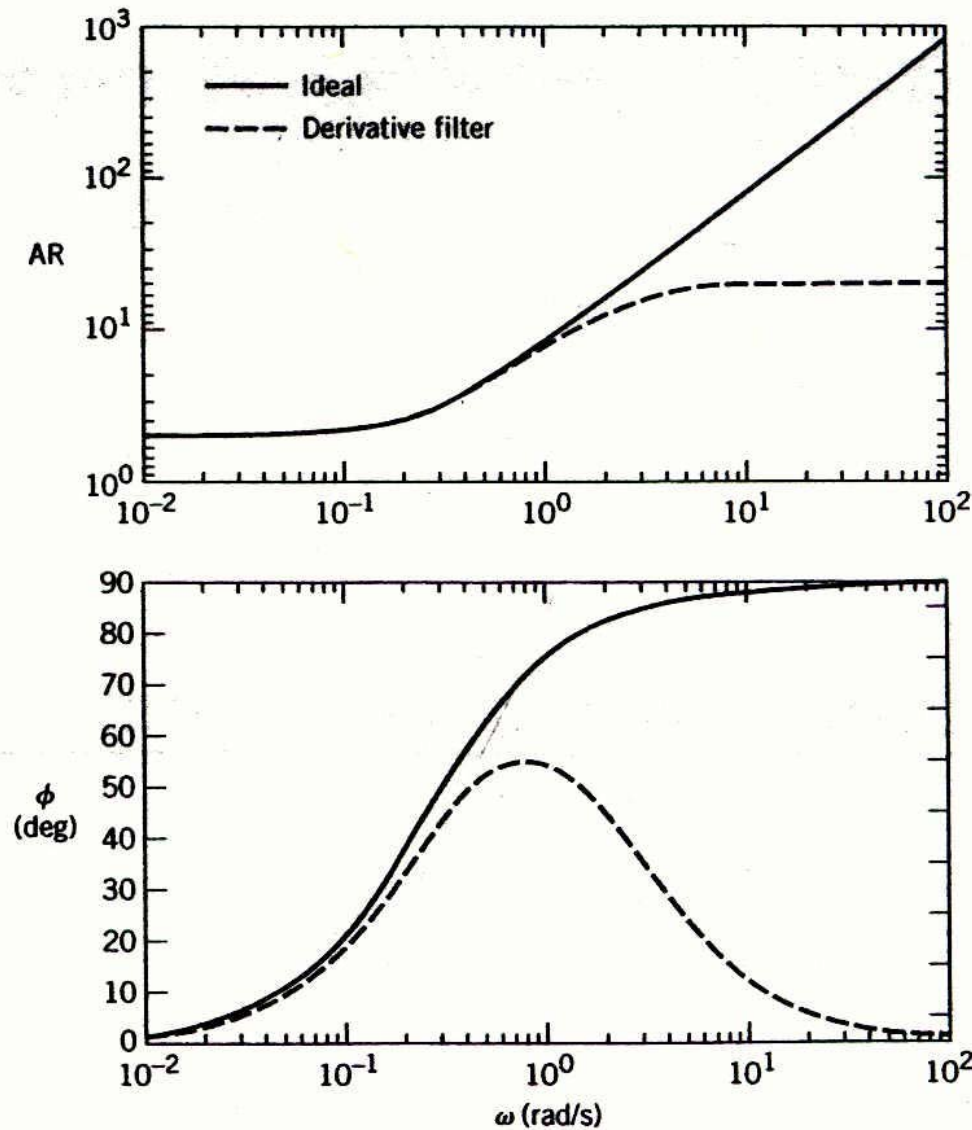


Figure 13.10 Bode plots of an ideal PD controller and a PD controller with derivative filter.

Idea: $G_c(s) = 2(4s + 1)$

With Derivative Filter:

$$G_c(s) = 2 \left(\frac{4s + 1}{0.4s + 1} \right)$$

PID Controller Forms

Parallel PID Controller. The simplest form in Ch. 8 is

$$G_c(s) = K_c \left(1 + \frac{1}{\tau_I s} + \tau_D s \right)$$

Series PID Controller. The simplest version of the series PID controller is

$$G_c(s) = K_c \left(\frac{\tau_I s + 1}{\tau_I s} \right) (\tau_D s + 1) \quad (13-73)$$

Series PID Controller with a Derivative Filter.

$$G_c(s) = K_c \left(\frac{\tau_I s + 1}{\tau_I s} \right) \left(\frac{\tau_D s + 1}{\alpha \tau_D s + 1} \right)$$

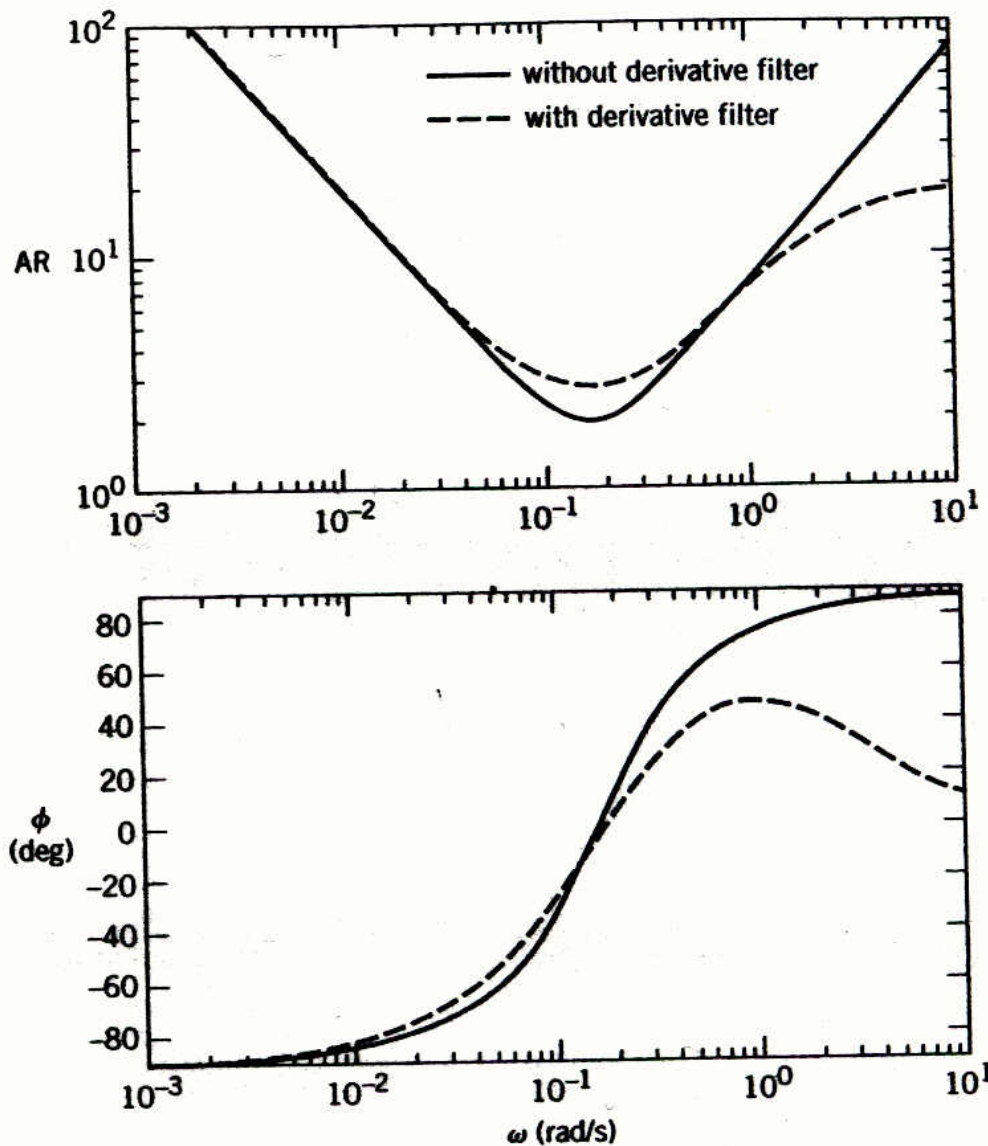


Figure 13.11 Bode plots of ideal parallel PID controller and series PID controller with derivative filter ($\alpha = 1$).

Idea parallel:

$$G_c(s) = 2 \left(1 + \frac{1}{10s} + 4s \right)$$

Series with
Derivative Filter:

$$G_c(s) = 2 \left(\frac{10s + 1}{10s} \right) \left(\frac{4s + 1}{0.4s + 1} \right)$$

Nyquist Diagrams

Consider the transfer function

$$G(s) = \frac{1}{2s + 1} \quad (13-76)$$

with

$$\text{AR} = |G(j\omega)| = \frac{1}{\sqrt{(2\omega)^2 + 1}} \quad (13-77a)$$

and

$$\varphi = \angle G(j\omega) = -\tan^{-1}(2\omega) \quad (13-77b)$$

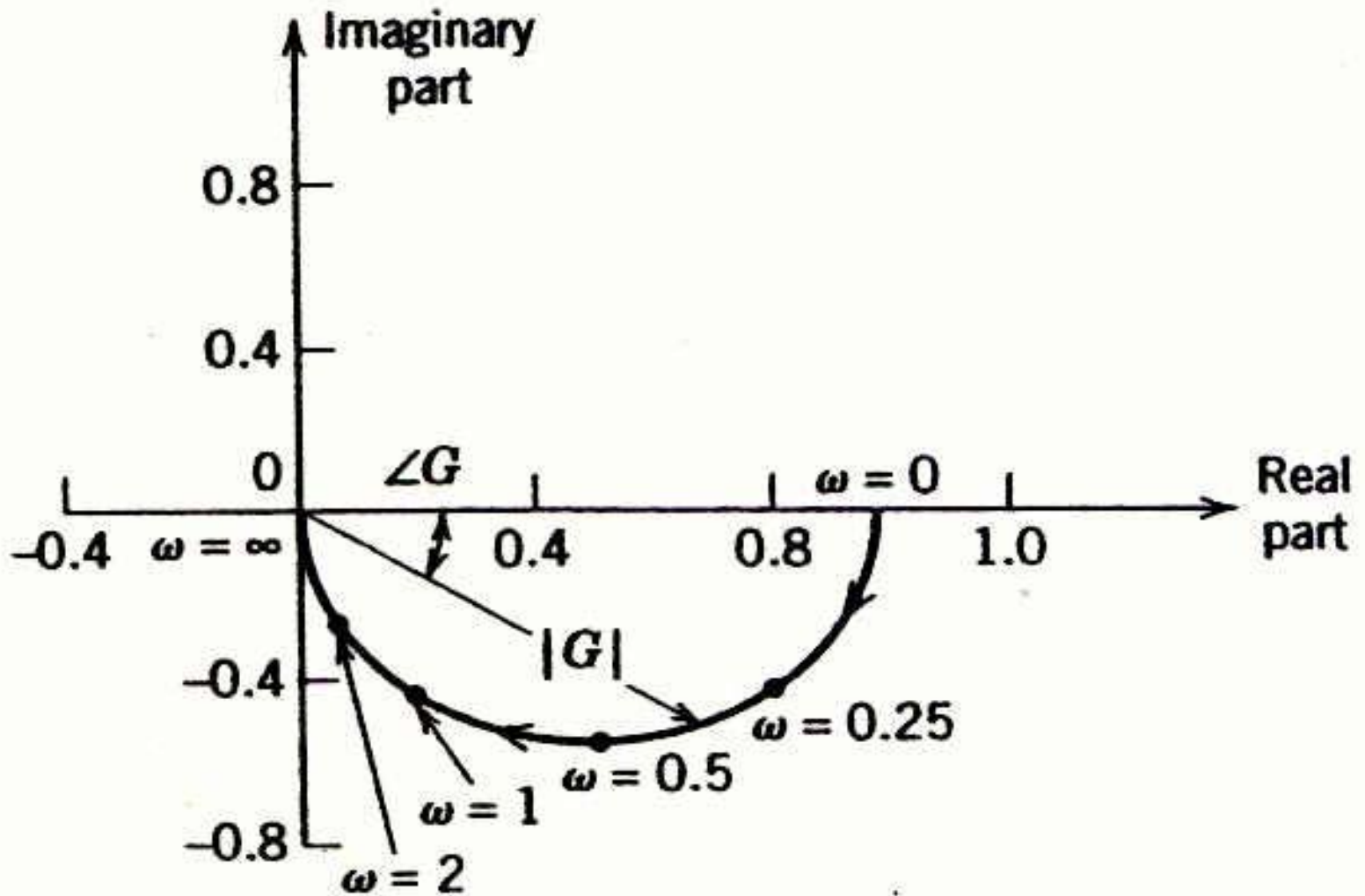


Figure 13.12 The Nyquist diagram for $G(s) = 1/(2s + 1)$ plotting $\text{Re}(G(j\omega))$ and $\text{Im}(G(j\omega))$.

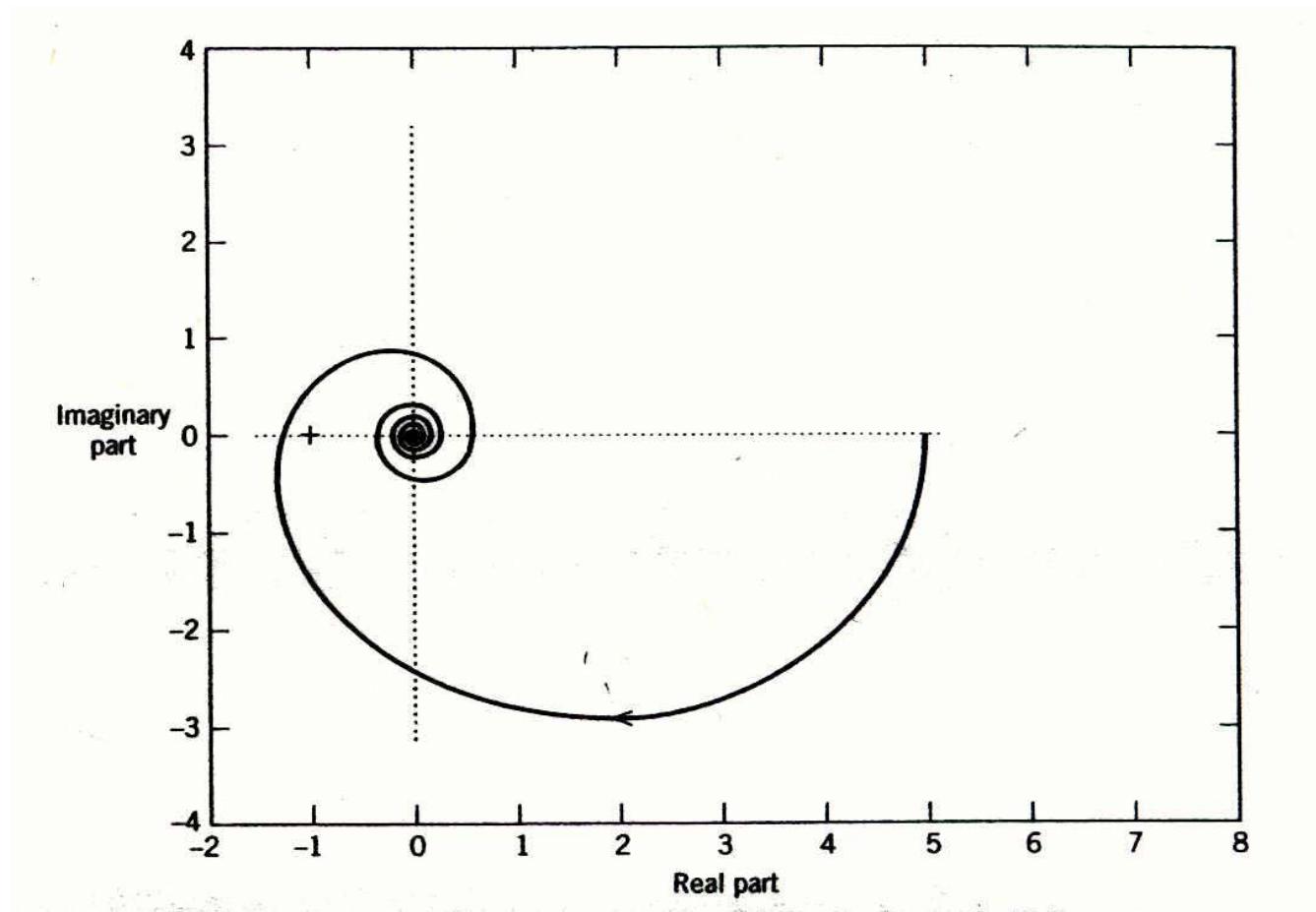


Figure 13.13 The Nyquist diagram for the transfer function in Example 13.5:

$$G(s) = \frac{5(8s + 1)e^{-6s}}{(20s + 1)(4s + 1)}$$