Analysis of current cycle error assisted iterative learning control for discrete nonlinear time-varying systems

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Abstract — Iterative learning control (ILC) is a simple and effective method for the control of systems that perform same task repetitively. In this paper, a current cycle error assisted ILC scheme for a class of discrete nonlinear time-varying systems is presented. Under a few reasonable assumptions, sufficient conditions for guaranteeing the convergence of the learning scheme are given and then proved by an inductive method. It is shown that the conditions are independent of the specific form of the state space model. Finally, some simulation results are given to illustrate the new control strategy.

Index terms — discrete nonlinear system; iterative learning control; convergence.

I. INTRODUCTION

Iterative learning control (ILC) is a simple and effective approach to the control of systems that are repetitive in nature. It was first introduced by Arimoto (1984), and since then many papers have developed various ILC schemes. Moore (1997) has surveyed on the ILC literature till 1997. Most ILC research has focused on continuous-time, dynamic systems. But for real-time implementation, all ILC schemes have to be designed in the discrete-time domain (Xu, 1997; Chien, 1998).

Next, some recent papers concerning about discrete-time ILC schemes. A discrete-time algorithm for discrete LTI systems without feed-through input was proposed in Saab (1995). He then extended the algorithm to discrete nonlinear time-varying systems with affine input action and linear output (Saab, 1999), the convergence rate can be exponential in $t^*$. Wang (1998) proposed an ILC scheme which uses only one-step training error with anticipation in time advance for off-line computation. His approach was developed for time-varying discrete dynamic systems with disturbances. Xu (1997) presented a P-type ILC algorithm in which the learning gain is a function of the system state and time. His algorithm is for valid systems with direct transmission from system inputs to outputs. Fang (1998) developed an ILC method for linear discrete-time multivariable systems which ensured that the desired trajectory is accurately tracked after only one learning iteration, if all of the model parameters are known.

As is well known, feedback control is the most commonly used method to eliminate huge overshoot in system output, it is also necessary to use this technology in ILC system. In ILC strategies, even though the learning gain matrices can be selected so that sufficient conditions for guaranteeing the convergence of the learning process are satisfied, it is possible that the tracking error can grow quite large before finally converging to zero (Hauser, 1987; Jang, 1995; Moon, 1998; Lee, 1997; etc.). This undesirable behavior can occur even when the applied learning algorithm has been proved to be exponentially convergent (Lee, 1997), because the learning control structure is basically open-loop (Jang,1995). Consequently, it is quite natural to employ the current cycle error in ILC scheme so as to eliminate the huge overshoot. But up to now, only a few results has been obtained for this type of ILC schemes. e.g.: Jang (1995) proposed a P-type learning law in a feedback configuration which can be applied to systems with relative degrees greater than one; Chien (1996) presented an ILC scheme using the concept of a forgetting factor and current error modification for a class of uncertain nonlinear, time-varying systems; Chen (1997) proposed a high-order P-type updating law in which more information of previous iterations are employed; Moon (1998) presents a frequency–domain design method to obtain an ILC law for LTI plant with multiplicative perturbations; Chien (1998) proposed an ILC scheme where feedforward learning algorithm is designed under a stabilizing controller, and its result is more general than Wang (1998). All of them is for continuous system except Chien (1998). Just like what pointed out in Chien (1998): in the discrete-time systems little work has been done on the performance improvement for ILC, which is certainly an important issue in practical applications.

In this paper we propose a current cycle error assisted discrete ILC scheme for a class of nonlinear discrete systems. Sufficient conditions for guaranteeing the convergence of the ILC scheme are given based on a few assumptions. Some simulation results are included to verify the results. The paper is organized as follows. In section 2, we state the ILC problem. Section 3 contains the main
results. Section 4 contains a few numerical examples to illustrate the new results. And finally, the conclusions appear in section 5.

II. PROBLEM FORMULATION

Consider the discrete, nonlinear, time-varying system described by the following state space model:

\[
\begin{aligned}
    x(i+1) &= f(x(i),u(i),i) \\
    y(i) &= c(x(i),i) + D(x(i),i)u(i)
\end{aligned}
\]

where state vector \( x(i) \in \mathbb{R}^n \), input vector \( u(i) \in \mathbb{R}^m \), output vector \( y(i) \in \mathbb{R}^{m_o} \), \( f(\bullet,\bullet,\bullet) : \mathbb{R}^n \times \mathbb{R}^n \times \{0,N\} \rightarrow \mathbb{R}^n \), \( c(\bullet,\bullet) : \mathbb{R}^n \times \{0,N\} \rightarrow \mathbb{R}^{m_o} \), and \( D(\bullet,\bullet) : \mathbb{R}^n \times \{0,N\} \rightarrow \mathbb{R}^{m_o \times n} \), for all \( i \in \{0,N\} \) (\( n \) is a positive integer). The system performs a given task repeatedly on the finite time interval \([0,N]\). In the \( k \)-th repetitive operation of the system, above equation becomes:

\[
\begin{aligned}
    x_k(i+1) &= f(x_k(i),u_k(i),i) \\
    y_k(i) &= c(x_k(i),i) + D(x_k(i),i)u_k(i)
\end{aligned}
\]

where subscript \( k \) denotes the iteration index. We want the system output \( y(i) \) to track a given trajectory \( y_d(i) \) for all \( i \in \{0,N\} \). To do this, we propose a new ILC law based on a combination of previous cycle information and feedback control based on the current cycle error, \( e_k \).

\[
\begin{aligned}
    u_k(i) &= u_{k-1}(i) + \sum_{j=0}^{i} L_{j,k-1}(x_{k-1}(i),i)e_{k-1}(i-j) \\
    &+ \sum_{j=0}^{i} B_{j,k}(x_k(i),i)e_k(i-j)
\end{aligned}
\]

where \( L_{j,k}(\bullet,\bullet,\bullet) : \mathbb{R}^n \times \{0,N\} \rightarrow \mathbb{R}^{m_o \times n} \) are bounded learning gain matrices, and \( e_k(i) = y_d(i) - y_k(i) \) is output tracking error. At first trial (i.e., \( k=0 \)), \( u_0(i) \) can be any value selected.

To get a convergence condition, we assume the following restrictions on the ILC system (2) - (3):

**Assumption 1.** The function \( f(\bullet,\bullet,\bullet) \) is continuous in \( x \) and \( u \), and the functions \( c(\bullet,\bullet) \) and \( D(\bullet,\bullet) \) are continuous in \( x \).

**Assumption 2.** For any realizable trajectory \( y_d(i) \), there exists a unique control input \( u_d(i) \) that can drive the system output to \( y_d(i) \).

**Assumption 3.** As \( k \rightarrow \infty \), the initial state for the \( k \)-th trial \( x_d(0) \) tends to the desired initial state \( x_d(0) \). i.e., \( \lim_{k \rightarrow \infty} x_k(0) = x_d(0) \).

**Assumption 4.** Matrix \([I + B_{j,k}(x_d(i),i)D(x_d(i),i)]\) is nonsingular for all \( k \) and \( i \), where \( I \) is the identity matrix with proper dimensions.

These assumptions are not overly restrictive for iterative learning control. Similar or more restrictive assumptions can be found in most papers about ILC (e.g.: Xu, 1997; Chien, 1998; etc.). Assumption 4 is needed because we use a current cycle error assisted ILC scheme. A key problem is: under what conditions the system (2) - (3) is convergent.

Through out this paper, the matrix norm is defined as the spectral radius of the matrix, i.e., the maximum absolute eigenvalue of the matrix; the vector norm is defined as the maximum absolute value of all elements in the vector.

III. ANALYSIS OF CONVERGENCE

**Lemma.** Suppose real series \( \{U_k\}_{k=0}^\infty \), \( \{V_k\}_{k=0}^\infty \) and \( \{W_k\}_{k=0}^\infty \) satisfy,

\[
\begin{aligned}
    a) &\quad U_{k+1} = W_k U_k + V_k \\
    b) &\quad \lim_{k \rightarrow \infty} V_k = 0 \\
    c) &\quad \| W_k \| \leq \rho < 1 \quad \text{for all } k > Q
\end{aligned}
\]

where \( U_k, V_k \in \mathbb{R}^r \) and \( W_k \in \mathbb{R}^{rd} \), \( Q \) is a finite positive integer. Then \( \lim_{k \rightarrow \infty} U_k = 0 \) is ensured.

**Proof:** Because \( \lim_{k \rightarrow \infty} V_k = 0 \), for arbitrary small positive real \( \varepsilon \), there exists integer \( P > 0 \), such that \( \| V_k \| < \varepsilon \) holds for all \( k > P \). Therefore,

\[
\| U_{k+1} \| = \| W_k U_k + V_k \| \leq \| W_k \| \| U_k \| + \| V_k \| \leq \rho \| U_k \| + \varepsilon
\]

for all \( k > \text{max}(P, Q) \). By equation (4), it is easy to show that following equation holds

\[
\lim_{k \rightarrow \infty} \| U_k \| \leq \frac{\varepsilon}{1 - \rho}
\]

Because \( \varepsilon \) is an arbitrarily small positive real number, so we have

\[
\lim_{k \rightarrow \infty} U_k = 0
\]

**Theorem.** Suppose that the control system described by
equations (2) (3) satisfies Assumptions 1 - 4. For any fixed positive integer \( M \), if \( L_{0,k}(x_{k}(i),i) \) and \( B_{0,k}(x_{k}(i),i) \) are chosen such that the following inequality holds for all \( i \in [0,N] \) and \( k > M \)

\[
\left\| \left[ I + B_{0,k+1}(x_{k+1}(i)i, i)D(x_{k+1}(i),i), i \right] \right\| < 1
\]

(7)

then \( x_{k}(i) \rightarrow x_{d}(i), u_{k}(i) \rightarrow u_{d}(i) \) and \( y_{k}(i) \rightarrow y_{d}(i) \) are ensured for all \( i \in [0,N] \) as \( k \rightarrow \infty \).

Proof Define three auxiliary functions as follows:

\[
\begin{align*}
A_{f}(x,u,i) &= f(x_{d}(i), u_{d}(i), i) \\
& - f(x_{d}(i), u_{d}(i), i) \\
A_{c}(x,i) &= c(x_{d}(i), i) - c(x_{d}(i), i) \\
A_{D}(x,i) &= D(x_{d}(i), i) - D(x_{d}(i), i)
\end{align*}
\]

(8)

By Assumption 1

\[
\begin{align*}
\lim_{x \to 0} A_{f}(x,u,i) &= 0 \\
\lim_{x \to 0} A_{c}(x,i) &= 0 \\
\lim_{x \to 0} A_{D}(x,i) &= 0
\end{align*}
\]

(9)

Define

\[
\begin{align*}
\Delta u_{k}(i) &= u_{d}(i) - u_{k}(i) \\
\Delta x_{k}(i) &= x_{d}(i) - x_{k}(i)
\end{align*}
\]

(10)

Combining equations (2), (3), (8) and (10) gives

\[
\begin{align*}
\Delta x_{k}(i + 1) &= x_{d}(i + 1) - x_{k}(i + 1) \\
& = f(x_{d}(i), u_{d}(i), i) - f(x_{d}(i), u_{d}(i), i) \\
& = A_{f}(\Delta x_{k}(i), i, \Delta u_{k}(i), i) \\
\Delta x_{k}(i) &= x_{d}(i) - x_{k}(i) \\
e_{k}(i) &= y_{d}(i) - y_{k}(i) \\
& = c(x_{d}(i), i) + D(x_{d}(i), i)u_{d}(i) \\
& - c(x_{d}(i), i) - D(x_{d}(i), i)u_{d}(i) \\
& = A_{c}(\Delta x_{k}(i), i) + A_{D}(\Delta x_{k}(i), i)u_{d}(i) \\
& + D(x_{d}(i), i)\Delta u_{k}(i)
\end{align*}
\]

(11)

From (13) and Assumption 4, we have

\[
\begin{align*}
\Delta u_{k+1}(i) &= \Delta u_{k}(i) - \sum_{j=0}^{i} L_{j,k}(x_{k}(i), i)e_{k}(i - j) \\
& - \sum_{j=0}^{i} B_{j,k+1}(x_{k+1}(i), i)e_{k+1}(i - j) \\
& = \Delta u_{k}(i) - \sum_{j=0}^{i} L_{j,k}(x_{k}(i), i)[A_{c}(\Delta x_{k}(i), i, i - j)] \\
& + A_{D}(\Delta x_{k+1}(i, i - j), i - j)u_{d}(i - j) \\
& + D(x_{d}(i, i - j), i - j)\Delta u_{k}(i - j) \\
& - \sum_{j=0}^{i} B_{j,k+1}(x_{k+1}(i), i)[A_{c}(\Delta x_{k+1}(i, i - j), i - j)] \\
& + A_{D}(\Delta x_{k+1}(i, i - j), i - j)u_{d}(i - j) \\
& + D(x_{d}(i, i - j), i - j)\Delta u_{k+1}(i, i - j)
\end{align*}
\]

(13)

By induction, we prove that the following equation holds for all \( i \in [0,1,2,...,N] \)

\[
\lim_{k \to \infty} \Delta x_{k}(i) = \lim_{k \to \infty} \Delta u_{k}(i) = \lim_{k \to \infty} e_{k}(i) = 0
\]

(15)

First, for \( i \in [0] \), i.e. \( i = 0 \), equation (14) becomes

\[
\begin{align*}
\Delta u_{k+1}(0) &= \Delta u_{k}(0) - \sum_{j=0}^{0} L_{j,k}(x_{k}(0), 0)e_{k}(0 - j) \\
& - \sum_{j=0}^{0} B_{j,k+1}(x_{k+1}(0), 0)e_{k+1}(0 - j) \\
& = \Delta u_{k}(0) - \sum_{j=0}^{0} L_{j,k}(x_{k}(0), 0)[A_{c}(\Delta x_{k}(0), 0, 0 - j)] \\
& + A_{D}(\Delta x_{k+1}(0, 0 - j), 0 - j)u_{d}(0 - j) \\
& + D(x_{d}(0, 0 - j), 0 - j)\Delta u_{k}(0 - j) \\
& - \sum_{j=0}^{0} B_{j,k+1}(x_{k+1}(0), 0)[A_{c}(\Delta x_{k+1}(0, 0 - j), 0 - j)] \\
& + A_{D}(\Delta x_{k+1}(0, 0 - j), 0 - j)u_{d}(0 - j) \\
& + D(x_{d}(0, 0 - j), 0 - j)\Delta u_{k+1}(0 - j)
\end{align*}
\]

(14)
\[ \delta u_{k+1}(0) = [I + B_{0,k+1}(x_{k+1}(0),0)]^{-1} \] 
\[ I - L_{0,k}(x_k(0),0) \delta x_k(0) \] 
\[ -[I + B_{0,k+1}(x_{k+1}(0),0)]^{-1} L_{0,k}(x_k(0),0) \delta u_k(0) \] 
\[ A_x(\delta x_k(0),0) + A_D(\delta x_k(0),0) u_d(0) \] 
\[ -[I + B_{0,k+1}(x_{k+1}(0),0)]^{-1} B_{0,k+1}(x_{k+1}(0),0) \delta u_k(0) \] 
\[ A_x(\delta x_{k+1}(0),0) + A_D(\delta x_{k+1}(0),0) u_d(0) \] 

(16)

By Assumption 3
\[ \lim_{k \to \infty} \delta x_k(0) = 0 \] 
(17)

From equations (9) and (17), we have
\[ \lim_{k \to \infty} A_x(\delta x_k(0),0) = 0 \]
\[ \lim_{k \to \infty} A_D(\delta x_k(0),0) = 0 \] 
(18)

By equations (16) and (18), inequality (7) and the lemma, it follows that
\[ \lim_{k \to \infty} \delta u_k(0) = 0 \] 
(19)

In view of Assumption 2, equation (19) implies that
\[ \lim_{k \to \infty} e_k(0) = 0 \] 
(20)

Thus, equation (15) holds for \( \forall i \in [0] \).

Suppose equation (15) holds for \( \forall i \in [0,1,\ldots,l] \), i.e.:
\[ \lim_{k \to \infty} \delta x_k(i) = \lim_{k \to \infty} \delta u_k(i) = \lim_{k \to \infty} e_k(i) = 0 \quad i=0,1,\ldots,l \] 
(21)

then to prove equation (15) also holds for \( \forall i \in [0,1,\ldots,l+1] \), we only need to prove that equation (15) holds for \( i = l+1 \). Combining equations (9), (11) and (21) gives
\[ \lim_{k \to \infty} \delta x_k(l+1) = \lim_{k \to \infty} A_x(\delta x_k(l),\delta u_k(l),l) = 0 \] 
(22)

From equations (9), (21) and (22), for \( \forall i \in [0,1,\ldots,l+1] \), we have
\[ \lim_{k \to \infty} A_x(\delta x_k(i),i) = 0 \]
\[ \lim_{k \to \infty} A_D(\delta x_k(i),i) = 0 \] 
(23)

It follows from equation (14), (21), (23), inequality (7) and the lemma that
\[ \lim_{k \to \infty} \delta u_k(l+1) = 0 \] 
(24)

Assumption 2 and equation (24) imply that
\[ \lim_{k \to \infty} e_k(l+1) = 0 \] 
(25)

Thus, equation (15) also holds when \( i = l+1 \). Combining (25) with equation (21), we know that equation (15) holds for \( \forall i \in [0,1,\ldots,l,l+1] \). Therefore, equation (15) holds for \( \forall i \in [0,1,2,\ldots,N] \).

*Remark 1.* The theorem above shows that the learning gains for the current cycle error have a direct influence on learning convergence. By employing these gains, learning convergence can be ensured even when the learning gains for the previous cycle error are not selected properly.

*Remark 2.* The system discussed here is more general than that of Xu (1997). By setting \( B_{j,k}(x_d(i),i) = 0 \) (for all \( j \) and \( k \)), and \( L_{j,k}(x_d(i),i) = 0 \) (for all \( j \neq 0 \) and \( k \)), our ILC scheme will become the same as in Xu (1997), but the convergence conditions given here are weaker, because the matrix norm defined here is smaller value than the norm defined by Xu (1997).

*Remark 3.* The theorem shows that learning convergence can be ensured even when the convergence conditions are not guaranteed during the previous \( M \) iterations. Here \( M \) is a finite positive integer.

*Remark 4.* The theorem shows that learning gains \( B_{j,k}(x_d(i),i) \) (\( j \neq 0 \)) and \( L_{j,k}(x_d(i),i) \) (\( j = 0 \)) have no influence on learning convergence if these gains are bounded. The learning convergence condition is independent of function \( f(\cdot,\cdot,\cdot) \) if the function satisfies Assumption 1.

### IV. SIMULATION RESULTS

In this section, a simple simulation example is presented.

Consider the robot manipulator given in Togai (1985) which is approximated by following continuous model
\[ \begin{bmatrix} \dot{\theta}(t) \\ \ddot{\theta}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -625 & -37 \end{bmatrix} \begin{bmatrix} \theta(t) \\ \dot{\theta}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 625 \end{bmatrix} u(t) \]

where \( \theta \) is the joint angle, and \( u \) is the input. To show the validity of the proposed ILC scheme for sampled systems in the presence of direct transmission, we select the system output to be \( y(t) = \dot{\theta}(t) \). Then the output equation can be written in the form of equation (1) after using a sample/hold mechanism. The desired output trajectory is
given by
\[ y_d(t) = 5\sin(2\pi t) \]
with operation period \( T = 14 \text{ s} \). Let \( x_1 = \theta \), \( x_2 = \dot{\theta} \), and the sampling period be \( T_s = 28 \text{ ms} \). Then the system model can be rewritten as (Togai, 1985):
\[
\begin{bmatrix}
  x_1(i+1) \\
  x_2(i+1)
\end{bmatrix} =
\begin{bmatrix}
  0.828 & 0.016 \\
  -10.044 & 0.234
\end{bmatrix}
\begin{bmatrix}
  x_1(i) \\
  x_2(i)
\end{bmatrix} +
\begin{bmatrix}
  0.172 \\
  10.044
\end{bmatrix} u(i)
\]
The system output equation and desired trajectory are
\[
y(i) = [-625 -37] \begin{bmatrix} x_1(i) \\ x_2(i) \end{bmatrix} + 625 u(i)
\]
\[
y_d(i) = 5\sin(2\pi T_s i) \quad i \in [0,500]
\]
For simplicity, a zero initial condition of the ILC system is used in the simulation.

Case 1. Let \( L_\alpha = 2 \), \( L_{i\alpha} = 1 \), \( L_{i\alpha} = 0 \) (for \( j > 2 \)), \( B_{0k} = 5 \), \( B_{i\alpha} = 1 \), \( B_{i\alpha} = 0 \) (for \( j > 2 \)). In this case, \( \| (I + B_{0k}D)^{-1}(I - L_\alpha D) \| = 0.3996 \) and the learning process is convergent according to our theorem. This conclusion is verified by the simulation results shown in Fig. 1. But without current cycle error learning, the system output error will become very large even after only one iteration, as shown in Fig. 2.

Case 2. Now change \( B_{0k} \) is changed to 50. In this case \( \| (I + B_{0k}D)^{-1}(I - L_\alpha D) \| = 0.04 \). Fig. 3 shows that the leaning process converges faster than for Fig. 1.

V. CONCLUSIONS

A current cycle error assisted ILC scheme has been developed for a class of discrete nonlinear time-varying systems. Both the system and the ILC scheme have very general forms. The inclusion of current cycle error feedback in the ILC scheme makes it possible to ensure learning convergence even when the learning gains for the previous cycle error are not selected properly. We show that the condition for guaranteeing the convergence of the learning is independent of the specific form of the state equation that describes the process. Finally, the proposed ILC scheme is applied to motion control of robot manipulator and simulation results verified the performance of the proposed ILC scheme.

Acknowledgements The first author appreciates the support from the American Zhu Kezhen Education Foundation, and the National Science Foundation of China (NSFC) under the project title ‘Open-closed-loop Iterative Learning Control of Nonlinear Systems (No.69874035)’.
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