Heat Transfer
Review of General Transport Equations

1. Consider an observer moving through space at a velocity *w*, which may be different than the fluid velocity *v*. The observer is continually measuring some fluid property designated by *S* (scalar) such as T, ρ, v_x, etc.

   What is the time rate of change of *S* as measured by observer:

$$\frac{dS}{dt} = \lim_{\Delta t \to 0}\left[\frac{S\big|_{t+\Delta t} - S\big|_t}{\Delta t}\right]$$

   In general, *S* is a function of *t, x, y, z* and the spatial coordinates are a function of time

$$\frac{dS}{dt} = \frac{d}{dt}\left[S\left(\left(x(t), y(t), z(t), t\right)\right]$$

   Using the chain rule, we can write with respect to fixed (*x, y, z*)

$$\frac{dS}{dt} = \left(\frac{dS}{dx}\right)_{y,z,t}\frac{dx}{dt} + \left(\frac{dS}{dy}\right)_{x,z,t}\left(\frac{dy}{dt}\right) + \left(\frac{dS}{dz}\right)_{x,y,t}\left(\frac{dx}{dt}\right) + \left(\frac{dS}{dt}\right)_{x,y,z} \quad \text{and}$$

   we see that

$$w_x \equiv \frac{dx}{dt}$$

$$w_y \equiv \frac{dy}{dt}$$

$$w_z \equiv \frac{dz}{dt}$$

   so

$$\frac{dS}{dt} = \left(\frac{\partial s}{\partial t}\right) + w_x\left(\frac{\partial s}{\partial x}\right) + w_y\left(\frac{\partial s}{\partial y}\right) + w_z\left(\frac{\partial s}{\partial z}\right) = \frac{\partial S}{\partial t} + \boldsymbol{w}\cdot\nabla S$$

   in the specific case the *w* = *v*, this is called the material derivative:

$$\frac{DS}{Dt} = \frac{\partial S}{\partial t} + \boldsymbol{v}\cdot\nabla S$$

If the observer is fixed in space $v = w = 0$ and

$$\frac{DS}{Dt} = \frac{\partial S}{\partial t}$$

With sufficient space, we can also show that

$$\frac{D}{Dt} \int_{Vol(t)} S dV = \underbrace{\int_{V_m(t)} \left(\frac{\partial S}{\partial t}\right) dV + \int_{A_m} S\, v \cdot n\, dA}$$

Reynold's Transport Theorem
a particular example of
Leibnig formula (page 732)

<u>Review of Transport Equations</u>

Material Derivative

$$\frac{DS}{Dt} = \frac{\partial S}{\partial t} + v \cdot \nabla S \qquad S \text{ — } Scalar$$

Conservation of Mass

$$\frac{D}{Dt} \int_V \rho dV = 0$$

and the Reynold's transport theorem states

$$\frac{D}{Dt} \int_V S dV = \int_V \left(\frac{\partial S}{\partial t}\right) dV + \cdot \int_A S\, \boldsymbol{v} \cdot \boldsymbol{n}\, dA$$

Divergence Theorem

$$\frac{D}{Dt} \int_V S dV = \int \left[\frac{\partial S}{\partial t} + \left(\nabla \cdot S\boldsymbol{v}\right)\right] dV, \text{ hence}$$

$$\frac{D}{Dt} \int_V \rho dV = \int \left[\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \boldsymbol{v}\right] dV$$

and arbitrary volumes implies

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \boldsymbol{v} = \boldsymbol{0} \rightarrow \text{ continuity equation}$$

equivalent to conversation of mass

Conservation of Linear Momentum

$$\{\text{time rate of change of linear momentum}\} = \{\text{the force on body}\}$$

$$\frac{D}{Dt}\int \rho \boldsymbol{v} dV = \int_{V_m} \underbrace{\int \rho \boldsymbol{g} dV}_{\text{body force}} + \underbrace{\int_A \boldsymbol{t}\boldsymbol{n}\, dA}_{\text{surface (shear) force}}$$

$$\boldsymbol{t}_n = \boldsymbol{T}\cdot\boldsymbol{n} \qquad \boldsymbol{T} \text{ — stress tensor}$$

and

$$\frac{D}{Dt}\int (\rho \boldsymbol{V})\, dV = \int_{V_m} \rho g dV + \int_{A_m} \boldsymbol{T}\cdot\boldsymbol{n}\, dA$$

Showing using Reynold's theorem, $\rho \boldsymbol{V} = S$

$$\frac{D}{Dt}\int \rho \boldsymbol{v} dV = \int \left[\frac{\partial(\rho \boldsymbol{v})}{\partial t} + (\nabla\cdot\rho \boldsymbol{v}\boldsymbol{v})\right] dV$$

$$= \frac{\partial \rho}{\partial t}\boldsymbol{v} + \boldsymbol{v}(\nabla\cdot\rho \boldsymbol{v}) + \rho\frac{\partial \boldsymbol{v}}{\partial t} + \rho \boldsymbol{v}\cdot\nabla \boldsymbol{v}$$

$$= \int \mathbf{v}\underbrace{\left[\frac{\partial \rho}{\partial t} + \nabla\cdot\rho \mathbf{v}\right]}_{0 \text{ by continuity equation}} + \rho\underbrace{\left[\frac{\partial \mathbf{v}}{\partial t} + \rho \mathbf{v}\cdot\nabla \mathbf{v}\right]}_{\frac{D\mathbf{v}}{Dt}}$$

$$\frac{D}{Dt}\int \rho \boldsymbol{v} dv = \int \rho\frac{D\boldsymbol{v}}{Dt} dV$$

$$\int_V \rho\frac{D\boldsymbol{v}}{Dt} = \int_V (\rho \boldsymbol{g} + \nabla\cdot\boldsymbol{T}) dV \quad \text{and}$$

arbitary volumes implies

$$\rho\frac{D\boldsymbol{v}}{Dt} - \rho \boldsymbol{g} - \nabla\cdot\boldsymbol{T} = 0$$

and we can also show that $\boldsymbol{T} = \boldsymbol{T}^T$ *or* $\boldsymbol{T}_{ij} = \boldsymbol{T}_{ji}$
The stress tensor can be written as

$T = -p\boldsymbol{u} + \boldsymbol{\tau}$	p	is isotropic pressure
	u	is unit tensor
	τ	is viscous stress tensor

$\boldsymbol{a}\cdot\boldsymbol{u} = \boldsymbol{u}\cdot\boldsymbol{a} = \boldsymbol{a}$ and

$\nabla\cdot p\boldsymbol{u} = \nabla p$

$$\nabla\cdot\left(pu + \tau\right) = -\nabla p + \nabla\cdot\boldsymbol{\tau}$$

$$\rho\frac{D\boldsymbol{v}}{Dt} = -\nabla p + p\boldsymbol{g} + \nabla\cdot\boldsymbol{\tau}$$

and for Newtonian fluids

$$\tau = \mu\left(\nabla\boldsymbol{v} + \nabla\boldsymbol{v}^T\right) + \underbrace{\boldsymbol{v}\left(\kappa - \frac{2}{3}\mu\right)\nabla\cdot\boldsymbol{v}}_{\text{usually very small}}$$

$$= \mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) + \delta_{ij}\left(\kappa - \frac{2}{3}\mu\right)\frac{\partial v_k}{\partial x_k}$$

for incompressible flow and constant velocity

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\rho\boldsymbol{v} = 0 \rightarrow \nabla\cdot\boldsymbol{v} = 0$$

and

$$\rho\left(\frac{\partial\boldsymbol{v}}{\partial t} + \boldsymbol{v}\cdot\nabla\boldsymbol{v}\right) = -\nabla p + \rho\boldsymbol{g} + \mu\nabla^2\boldsymbol{v}$$

is the Navier-Stokes equation.

Finally, conservation of energy, we showed earlier that

$$\frac{D}{Dt}\int\rho\left(e + \frac{1}{2}v^2\right)dV = \int -\boldsymbol{q}\cdot n\ dA + \int \boldsymbol{t_n}\cdot\boldsymbol{v}\ dA$$

$$+\int\rho\boldsymbol{g}\cdot\boldsymbol{v}\ dV + \int\Phi dV$$

and doing similar tricks we can show

$$\frac{D}{Dt}\int\rho\left(e + \frac{1}{2}v^2\right)dV = \int\rho\frac{D}{Dt}\left(e + \frac{1}{2}v^2\right)dV$$

$$\int -\boldsymbol{q}\cdot n dA = \int -\nabla\cdot\boldsymbol{q} dV$$

$$\int \boldsymbol{t}_n\cdot\boldsymbol{v}\ dA = \int \nabla\cdot(\boldsymbol{T}\cdot\boldsymbol{v})\ dV$$

and hence

$$0 = \int\left[\rho\frac{D}{Dt}\left(e+\frac{1}{2}v^2\right)+\nabla\cdot\boldsymbol{q}-\nabla\cdot(\boldsymbol{T}\cdot\boldsymbol{v})-\rho\boldsymbol{g}\cdot\boldsymbol{v}-\Phi\right]dV$$

and $\rho\frac{D}{Dt}\left(e+\frac{1}{2}v^2\right)=-\nabla\cdot\boldsymbol{q}+\nabla\cdot(\boldsymbol{T}\cdot\boldsymbol{v})-\rho\boldsymbol{g}\cdot\boldsymbol{v}-\Phi$

is the general energy balance

We have already used the Steady State case with zero velocity

$-\nabla\cdot\boldsymbol{q}+\Phi=0$ and $\boldsymbol{q}=-k\nabla T$ to get

$\nabla^2 T+\Phi=0$ as the thermal equation.

Now we need to retain v for times when convection is important

$$\rho\frac{\partial}{\partial t}\left(e+\frac{1}{2}v^2\right)+v\cdot\nabla\left(e+\frac{1}{2}v^2\right)=-\nabla\cdot\boldsymbol{q}+\nabla(\boldsymbol{T}\cdot\boldsymbol{v})+\rho\boldsymbol{g}\cdot\boldsymbol{v}-\Phi$$

(for v constant = 0 we had) $\rho\frac{\partial e}{\partial t}=-\nabla\cdot\boldsymbol{q}+\Phi$

and $\frac{\partial T}{\partial t}=+\alpha\nabla^2 T+\Phi/\rho Cp$

To simplify these equations, examine $v\cdot$ stress equation of motion

$$\rho\frac{D\boldsymbol{v}}{Dt}=\rho\boldsymbol{g}+\nabla\cdot\boldsymbol{T},\ \text{dot } \boldsymbol{v}$$

$$\rho\boldsymbol{v}\cdot\frac{D\boldsymbol{v}}{Dt}=\boldsymbol{v}\cdot\rho\boldsymbol{g}+\boldsymbol{v}\cdot\nabla\cdot\boldsymbol{T}$$

$$\rho\boldsymbol{v}\cdot\frac{D\boldsymbol{v}}{Dt}=\rho\frac{D\frac{1}{2}\boldsymbol{v}\cdot\boldsymbol{v}}{Dt}=\rho\frac{D}{Dt}\left(\frac{1}{2}v^2\right)$$

and

$\nabla\cdot(\boldsymbol{T}\cdot\boldsymbol{v})=\nabla\cdot(\boldsymbol{v}\cdot\boldsymbol{T})$ T is symmetric

an identity from tensors gives

$$\nabla\cdot\boldsymbol{v}\cdot\boldsymbol{T}=\boldsymbol{v}\cdot(\nabla\cdot\boldsymbol{T})+\cdot\nabla\boldsymbol{v}:\boldsymbol{T}$$

Mechanical engineering equation subtract from above general energy equation

$$\rho \frac{D}{Dt}\left(\frac{1}{2}v^2\right) = \rho \boldsymbol{g}\cdot\boldsymbol{v} + \nabla\left(\boldsymbol{v}\cdot\boldsymbol{T}\right) - \nabla\boldsymbol{v}:\boldsymbol{T}$$

to get

$$\rho \frac{De}{Dt} = -\nabla\cdot\boldsymbol{q} + \nabla\boldsymbol{v}:\boldsymbol{T} + \Phi$$

and using $\boldsymbol{T} = \tau + p\boldsymbol{u}$

$$\rho \frac{De}{Dt} = -\nabla\cdot\boldsymbol{q} - p\nabla\cdot\boldsymbol{v} + \nabla\boldsymbol{v}:\boldsymbol{\tau} + \Phi$$

and with $\boldsymbol{q} = -k\nabla T$

$$\rho \frac{De}{Dt} = \nabla\left(+k\ \nabla T\right) - p\nabla\cdot\boldsymbol{v} + \nabla\boldsymbol{v}:\boldsymbol{\tau} + \Phi$$

Let's examine some basic cases

A) Constant volume — constant density

$\nabla\cdot\boldsymbol{v} = 0$ from continuity

$$\rho \frac{De}{Dt} = \nabla\left(k\ \nabla T\right) + \nabla\boldsymbol{v}:\tau + \Phi$$

$$\frac{De}{Dt} = \left(\frac{\partial e}{\partial T}\right)_v \frac{DT}{Dt} + \left(\frac{\partial e}{\partial \rho}\right)_T \frac{D\rho}{Dt}$$

$\left(\frac{\partial e}{\partial T}\right)_V = C_V$ and $\left(\frac{D\rho}{Dt}\right) = 0$ Since ρ is constant

$$\rho C_v \frac{DT}{Dt} = \nabla\cdot\left(k\ \nabla T\right) + \nabla\boldsymbol{v}:\boldsymbol{\tau} + \Phi$$

$C_v \approx C_p$ for many solid and liquid materials

B) Constant pressure

$\frac{D\rho}{Dt} + \rho\nabla\cdot v = 0$ and $p\nabla\cdot\boldsymbol{v} = \frac{p}{\rho}\frac{D\rho}{Dt}$

$$\rho \frac{De}{Dt} = \nabla\left(k\ \nabla T\right) + \frac{p}{\rho}\frac{D\rho}{Dt} + \nabla\boldsymbol{v}:\boldsymbol{\tau} + \Phi$$

$$\frac{p}{\rho}\frac{D\rho}{Dt} = p\rho\frac{D}{Dt}\left(\frac{1}{\rho}\right) = -\rho\frac{D}{Dt}\left(\frac{p}{\rho}\right)$$

and $\rho \frac{D}{Dt}\left(e + \frac{p}{\rho}\right) = \nabla\left(k\nabla T\right) + \nabla\boldsymbol{v}:\boldsymbol{\tau} + \Phi$

$$e + pV = H = e + \frac{p}{\rho}$$

and

$$\frac{DH}{Dt} = \left(\frac{\partial H}{\partial T}\right)_p \frac{DT}{Dt} + \left(\frac{\partial H}{\partial P}\right)_T \frac{DP}{Dt} = C_p \frac{DT}{Dt}$$

$$\rho C_p \frac{DT}{Dt} = \nabla \cdot \left(k \ \nabla T\right) + \nabla \boldsymbol{v} : \boldsymbol{\tau} + \Phi$$ — constant P

In general, P and V are not constant and we get

$$\rho C_p \frac{DT}{Dt} = \nabla \cdot \left(k \ \nabla T\right) + T\beta \frac{Dp}{Dt} + \nabla \boldsymbol{v} : \boldsymbol{\tau} + \Phi$$

$$\beta = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_p$$ — coefficient of thermal expansion

- - important to free convection

Now we need to examine each term above to find out if it is important.

Order of magnitude — viscous dissipation $\rho C_p \frac{DT}{Dt} = \nabla v : \tau$

no generation, steady state, no conduction

$\nabla v : \tau$ — energy lost from fluid momentum by friction.

Examine laminar flow in a tube, $v_z = v_z$ (r) only

$$\nabla v : \tau = \mu \left(\frac{\partial v_z}{\partial r}\right)^2$$

For a tube $\quad v_z\left(r\right) = \frac{\Delta P R^2}{4\mu L}\left[1 - \left(\frac{r}{R}\right)^2\right]$

$$V_z\left(r\right) = 2 v_{z_{av}} \left[1 - \left(\frac{r}{R}\right)^2\right], \quad v_{z_{av}} = \frac{\Delta P R^2}{8\mu L} = \left\langle v_z \right\rangle$$

$$\frac{\partial v_z}{\partial r} \le \frac{V_{z(\max)}}{R} \sim \frac{4\left\langle V_z \right\rangle}{D}$$

$$\left(\frac{\partial v_z}{\partial r}\right)^2 \approx \frac{16\left\langle V_z \right\rangle}{D^2}$$

$$\nabla v : \tau \approx \frac{16\mu \left\langle V_z \right\rangle^2}{D^2}$$

$\frac{DT}{Dt} = v_z \frac{\partial T}{\partial z}$ for steady state

and $v_z \frac{\partial T}{\partial z} = \rho Cp \cdot \frac{DT}{Dt} = \nabla v : \tau = \frac{16\mu \langle v_z \rangle^2}{D^2}$

Solving for $\frac{DT}{\partial z}$ we get

$$\frac{16\mu \langle v_z \rangle}{\rho C_p D^2} = \frac{\partial T}{\partial z}$$

in terms of Reynolds' number $D\langle v_z \rangle \rho \big/ \mu = \frac{D\langle v_z \rangle}{\upsilon} = \text{Re}$ $\qquad \frac{\nu \, \text{Re}}{D} = \langle v_z \rangle$

$$\frac{16\mu \frac{\nu \, \text{Re}}{D}}{\rho C_p D^2} = \frac{\partial T}{\partial z}$$

$$\frac{16\nu^2 \, \text{Re}}{C_p D^3} \cong \frac{\partial T}{\partial z}$$

$\text{Re} \sim 10^3$

$\nu = .2 \; cm^2 / \text{sec}$

for air $D \sim 1 \; cm$

$C_p \sim .2$ cal/gm°C

$\frac{\partial T}{\partial z} \tilde{\approx} 1x10^{-4\circ} C/cm$ for air--negligible

for water

$N_{\text{Re}} \sim 10^3$

$v = 10^{-2} \; cm^2/\text{sec}$

$D \sim 1cm$

$C_p \sim 1 cal/gm°C$

$\frac{\partial T}{\partial z} \approx 10^{-8\circ} C/cm$

for heavy oil

$N_{\text{Re}} \sim 10^3 \;\; v \sim 10 \, cm^2/\text{sec}$

$D \sim 1cm \;\; Cp \sim 0.5 \, cal/gm°C$

$\frac{\partial T}{\partial z} \sim .1° C/cm$ --too big to ignore?

Reversible work — laminar tube flow

$$\frac{DT}{Dt} = \left(T\beta \frac{Dp}{Dt} \Big/ \rho Cp \right)$$

steady flow $\qquad \dfrac{DT}{Dt} = v_z \dfrac{\partial T}{\partial z}$

$$\frac{Dp}{Dt} = v_z \frac{\partial p}{\partial z}$$

$$\frac{\partial T}{dz\left(_{r.w.}\right)} = \left(T\beta \frac{\partial p}{\partial z} \Big/ \rho Cp \right)$$

$$\frac{\partial T}{dz\left(_{r.w.}\right)} = \left(\frac{T\beta}{\rho Cp} \right) \frac{\partial p}{\partial z}$$

$$\langle v_z \rangle = \frac{\Delta P}{L} \frac{R^2}{8_\mu} = \frac{dP}{dz} \frac{R^2}{8_\mu}$$

$$\frac{8_\mu \langle v_z \rangle}{R^2} = \frac{dP}{dz}$$

$$\frac{32_\mu \langle v_z \rangle}{D^2} = \frac{dP}{dz}$$

$$\langle v_z \rangle = \frac{\nu \, \mathrm{Re}}{D}$$

$$\frac{32 \rho \nu^2 \, \mathrm{Re}}{D^3} = \frac{dP}{dz} -$$

$$\frac{\partial T}{dz\left(_{r.w.}\right)} \cong 32 T\beta \frac{\nu^2 \, \mathrm{Re}}{C_p D^3}$$

$$\approx 2 \, \mathrm{T}\beta \left(\frac{\partial T}{\partial z} \right)_{(\text{viscous dissipation})}$$

for air (ideal gas) $\beta = \dfrac{1}{T}$ and

$\dfrac{\partial T}{\partial z_{work}} \sim \dfrac{\partial T}{\partial z}_{\textit{viscous dissipation}}$ for IDEAL

for water $\beta \sim 10^3 / {}^\circ C, \;\; 32 \, \beta \, \mathrm{T} \sim 10$ and

$$\frac{\partial T}{\partial z_{work}} \sim 10 \frac{\partial T}{\partial z}_{\textit{viscous dissipation}}$$