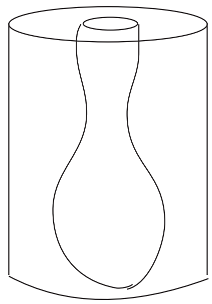
Heat Transfer — Connection to Thermodynamics



Thermos – How much heat is removed from 1 liter of water as it cools from 90°C to 80°C?

This is a question that we can answer via thermodynamics.
Energy conservation states that

energy in – energy out = net change in energy of system

$$U_i - U_f = \Delta U$$

$$H_i - H_f = \Delta H$$

and from definitions of heat capacity

$$dU_v = mC_v dT_f \qquad dU_p = mC_p dT$$

$$\Delta U = m\int_{T_i}^{T_f} C_v dT \qquad \Delta H = m\int_{T_i}^{T_f} C_p dT$$

which for constant C_v, C_p gives

$$\Delta U = mC_v\left(T_f - T_i\right) \qquad \Delta H = mC_p\left(T_f - T_i\right)$$

Values of C_v, C_p depend on the material.

For an ideal gas

$$C_p = C_v + R$$

Most liquids and solids, under normal conditions, are incompressible, or have a constant density as a function of pressure. This means that constant volume and constant pressure changes are the same and

$$C_p = C_v$$

and $\Delta U = C_v dT = \Delta H = C_p dT$

Often the heat capacity of a material changes with temperature; hence, we often need an average C_p for calculations. From the 1st Law, $\Delta U = Q + W$, and mainly we will be dealing with cases where $W = 0$. However, back to the thermos. We know that eventually the hot drink in the thermos will cool off. Thermodynamics will not distinguish between a good thermos that takes a long time to cool off and a poor thermos. What is important is the heat transfer rate – not just Q, but $\dot{Q}$ to be labeled q or Q later today – somewhat confusing), the rate of heat/time being exchanged.

$$Q = \int_0^{\Delta t} \dot{Q}\, dt$$

Once we know $\dot{Q}(t)$ we can calculate Q if $\dot{Q}$ is constant

$$Q = \dot{Q}\,\Delta t$$

The rate of heat transfer per unit area normal to the direction of the heat transfer is heat flux

$$\dot{q} = \frac{\dot{Q}}{A} \qquad (\text{W/m}^2)$$

in which *A* is the heat transfer area. In English, watts $\dot{q}$ is Btu/hr ft^2. The heat flux can vary with time, temperature, and position.

Example 1:

A 10 cm diameter copper ball is to be heated from 100°C to 150°C in 30 minutes. $\rho_{Cu} = 8950$ kg/m^3 and $C_p = 0.395$ kJ/kg°C. What is: a) the total amount of heat transfer to the ball, b) the average heat transfer rate, and c) the average heat flux at the ball surface?

a) The total energy requirement is from thermodynamics

$$\Delta U \cong \Delta H = Q = mC_p\left(T_f - T_i\right)$$

$$= \rho \mathrm{V} C_p\left(T_f - T_i\right) = \rho \mathrm{V} C_p \frac{4}{3}\pi r^3\left(T_f - T_i\right)$$

$$m = \rho\,\mathrm{v} = \rho\frac{4}{3}\pi r^3 = \frac{\pi}{6}\rho D^3 = \left(\frac{\pi}{6}\right)(8950)(.1\,\mathrm{m})^3 = 4.7\,\mathrm{kg}$$

$$Q = (4.7\ \mathrm{kg})\left(0.395\frac{\mathrm{kJ}}{\mathrm{kg}}{}^\circ\mathrm{C}\right)(150-100^\circ\mathrm{C}) = 92.6\ \mathrm{kJ}$$

b) While the average rate of heat transfer is

$$\dot{Q}_{\text{ave}} = \frac{Q}{\Delta t} = \frac{92.6\ \mathrm{kJ}}{1800\ \mathrm{s}} = -.0514\frac{\mathrm{kJ}}{\mathrm{s}} = 51.4\,W$$

c) The heat flux is defined as the heat transfer/time · area

$$q_{\text{ave}} = \frac{\dot{Q}_{\text{ave}}}{A} = \frac{\dot{Q}_{\text{ave}}}{\pi D^2} = \frac{51.4\,W}{\pi(.1)^2} = 1640\,W/\mathrm{m}^2$$

These are average values of $\dot{Q}$ and q are likely not to be equal to the actual instantaneous value $\dot{Q}$ or q during the process.

Example 2: Heat loss from ducts in basement

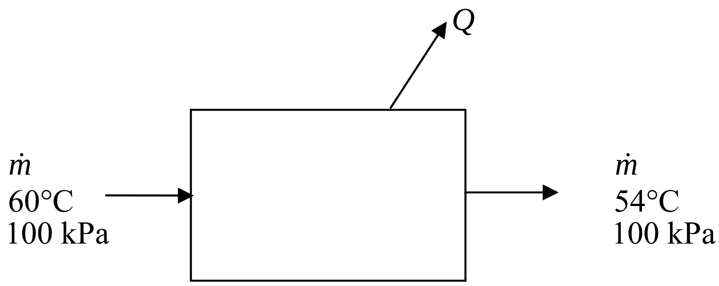
A 5 m long section of an air heating system of a house passes through an unheated basement. The duct is 20 x 25 cm. Hot air enters the duct at 200 kPa and 60°C at an average velocity of 5 m/s. The temperature of the air drops to 54°C through the loss of heat to the basement. What is the rate of heat loss?

A simple energy balance shows

Energy in – Energy out = Energy accumulated at steady conditions, nothing accumulates in the duct (accumulation refers to things changing in time) so

Energy in = Energy out

$$mC_p\left(T_{\text{in}} - T_{\text{ref}}\right) = mC_p\left(T_{\text{out}} - T_{\text{ref}}\right) + \dot{Q}$$



$$\dot{m}\ C_p\left(T_{\text{in}} - T_{\text{out}}\right) = \dot{Q}$$

$$C_p = 1.0\ \text{KJ/kg°C}$$

$$\rho = \frac{p}{RT} = \frac{100\ \text{kPa}}{\left(.287\frac{\text{kPa}\cdot\text{m}^3}{\text{kg}\cdot\text{K}}\right)\left(60+273\right)\text{K}} = 1.05\ \frac{\text{kg}}{\text{m}^3}$$

Area of duct = (.20 m) (.25 m) = .05 m^2

$$\dot{m} = \rho\,\text{v}\,A = \left(1.05\frac{\text{kg}}{\text{m}^3}\right)\cdot\left(5\ \text{m/s}\right)\cdot\left(.05\ \text{m}^2\right) = .26\ k\,\text{g/s}$$

$$\dot{Q} = \left(.26\frac{\text{kg}}{\text{s}}\right)\cdot\left(1.0\frac{\text{kJ}}{\text{kg°C}}\right)\cdot\left(60-54\,°\text{C}\right)$$

$$\dot{Q} = 1.6\ \frac{\text{kJ}}{\text{s}} \equiv 1.6\ \text{kW}$$

If this energy costs money, -- say, natural gas is \$.60/therm (1 therm = 105, 500 KJ)

$$\text{Cost of loss} = \text{Rate of heat loss}\cdot\frac{\text{Cost of heat}}{\text{Unit of heat}} = \text{Rate of heat loss}\ \cdot\ \frac{\$.60}{105{,}500\ \text{KJ}}$$

$$= \frac{5760\ \text{KJ}}{\text{hr}}\ \cdot\ \frac{\$.60}{105{,}500\ \text{KJ}} = \$.032/\text{hr}$$

We have discussed heat transfer, but not mechanisms. How does heat move from one place to another? What controls the rates? How do we enhance or impede heat transfer?

Heat Transfer – The Quick and Dirty

Conduction: When a temperature gradient exists within a body, there is an energy transfer from the hot region to the cooler region. The energy is said to be transferred by "conduction" and the rate per unit area is proportional to the gradient along the direction of heat transfer:

$$q = \frac{Q}{A} \sim \frac{dT}{dx} \text{ in } x \text{ direction}$$

The constant of proportionality, *k,* is called the thermal conductivity of the material and is always positive

q is the rate of energy (heat) transfer, the negative sign is inserted so that heat is transferred from hot to cold. This is known as Fourier's Law after the French mathematical physicist, Joseph Fourier.

Heat flow in watts, $k - \frac{\text{watts}}{m \cdot K}$

$$Q \sim -kA\frac{\Delta T}{\Delta x}$$

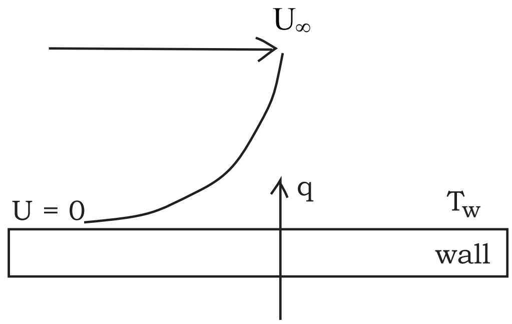
Table 1 Thermal Conductivity of Various Materials at 0°C

Material	W/m °C	Btu/ft h °F
Metals:		
Silver (pure)	410	237
Copper (pure)	385	223
Aluminum (pure)	202	117
Nickel (pure)	93	54
Iron (pure)	73	42
Carbon Steel, 1% C	43	25
Lead (pure)	35	20.3
Chrome-nickel steel (18% Cr. 8% Ni)	16.3	9.4
Nonmetallic solids:		
Diamond	2300	1329
Quartz, parallel to axis	41.6	24
Magnesite	4.15	2.4
Marble	2.08-2.94	1.2-1.7
Sandstone	1.83	1.06
Glass, window	0.78	0.45
Maple or oak	0.17	0.096
Hard rubber	0.15	0.087
Polyvinyl chloride	0.09	0.052
Styrofoam	0.033	0.019
Sawdust	0.059	0.034
Glass wool	0.038	0.022
Ice	2.22	1.28
Liquids:		
Mercury	8.21	4.74
Water	0.556	0.327
Ammonia	0.540	0.312
Lubricating oil, SAE 50	0.147	0.085
Freon 12, CCl_2F_2	0.073	0.042
Gases:		
Hydrogen	0.175	0.101
Helium	0.141	0.081
Air	0.024	0.0139
Water vapor (saturated)	0.0206	0.0119
Carbon dioxide	0.0146	0.00844

Convection

A hot plate of metal will cool faster when placed in front of a fan than in still air. We say that the heat is convected away. The velocity of the air influences the rate; however, it is found that doubling the air velocity does not double the rate of heat transfer. These details will be dealt with later.

Free Stream



The velocity of the stream will be reduced to zero at the wall, so very near the wall, the heat transfer is by conduction. However, the thickness of this layer of slow moving fluid depends on the velocity of the free stream, and hence, $\Delta T / \Delta \sim \frac{T_w - T_\infty}{\Delta x}$ depends on the fluid velocity through Δx. So in reality, convection is a form of conduction. In most calculations we use a simple model:

$$Q = hA\left(T_w - T_\infty\right)$$

h is called the heat transfer coefficient in $\frac{\text{W}}{m^2 K}$ units. h depends on geometry, the fluid being convected, somewhat on the solid surface, etc.

We divide convection into 2 types depending on whether the flow is imposed externally or by the temperature difference. A hot or cold surface in contact with a fluid can induce a flow by causing changes in density. This can cause flow and increase the rate of heat transfer- “free” vs “forced” convection.

Table 2 Approximate Values of Convection Heat-Transfer Coefficients

Mode	h $W/m^2 \cdot °C$	h $\frac{Btu}{ft^2 \ h \cdot °F}$
Free convection, $\Delta T = 30°C$		
Vertical pate 0.3 m [1 ft] high in air	4.5	0.79
Horizontal cylinder 5-cm diameter, in air	6.5	1.14
Horizontal cylinder, 2-cm diameter in water	890	157
Heat transfer across 1.5-cm vertical air gap with $\Delta T = 60°C$	2.64	0.46
Fine wire in air d=0.02 mm, $\Delta T = 55°C$	490	86
Forced convection		
Airflow at 2 m/s over 0.2-m square plate	12	2.1
Airflow at 35 m/s over 0.75-m square plate	75	13.2
Airflow at Mach number = 3, p = 1/20 atm, $T ? = -40°C$ across 0.2-m square plate	56	9.9
Air at 2 atm flowing in 2.5-cm-dioameter tube at 10 m/s	65	11.4
Water at 0.5 kg/s flowing in 2.5-cm-diameter tube	3500	616
Airflow across 5-cm-diameter cylinder with velocity of 50 m/s	180	32
Liquid bismuth at 4.5 kg/s and 420°C in 5.0-cm-diameter tube	3410	600
Airflow at 50 m/s across fine wire, d= 0.04 mm	3850	678
Boiling Water		
In a pool or container	2500-35.000	440-6200
Flowing in a tube	5000-100.000	880-17.600
Condensation of water vapor, 1 atm		
Vertical Surfaces	4000-11.300	700-2000
Outside horizontal tubes	9500-25.000	1700-4400

Thermal Radiation

An ideal material called a "black body" emits energy at a rate proportional to the fourth power of the temperature:

$$Q = \sigma A T^4$$

σ - Stefan-Boltzmann Constant = $5.669 \cdot 10^{-8} \ \frac{\text{w}}{m^2 K^4}$

The net radiant exchange between 2 black bodies is

$$\frac{Q}{A} = q \sim \sigma\left(T_1^4 - T_2^4\right)$$

Dark materials, cavities, space approximate a black body. Other surfaces – like reflective objects – shiny metals absorb and emit less – are called grey.

You must also take into account that not all the radiation leaving one surface will reach the second surface since radiation travels in straight lines. Some is lost.

$$Q = F_e F_G \, \sigma A\left(T_1^4 - T_2^4\right)$$

F_e in an emissivity function – depends on wavelength and F_G is a view factor function. If one object is enclosed by another then $F_G \to 1$ and if F_e is simply the emissivity, we have

$Q = \in \sigma A\left(T_1^4 - T_2^4\right)$

Examples:

Conduction through copper plate:

$$k_{CU} = 370 \ W/m°C$$

One face at 400°C, the other at 100°C, 3 cm thick

$$\frac{Q}{A} = -k \ dT/dx = q = \text{constant}$$

integrating for constant q gives (show details later)

$$q = -K \ \Delta T/\Delta x = \frac{-370\left(W/m\,°C\right)\left(100-400\right)\left(°C\right)}{3\cdot 10^{-2}\left(\text{m}\right)}$$

$$q = 3.7 \ MW/m^2$$

Convection: Air@ 20°C blows over a hot plate

50 x 75 cm^2 @ 250°C $\qquad h = 25\,W/m^2\,°C \qquad$ What is Q?

$$Q = hA\left(T_w - T_\infty\right)$$

$$= \left(25\frac{w}{m^2\,°C}\right)\left(.5\ m\right)\left(.75\ m\right)\left(250-20°C\right)$$

$$Q = 2.156\ kW$$

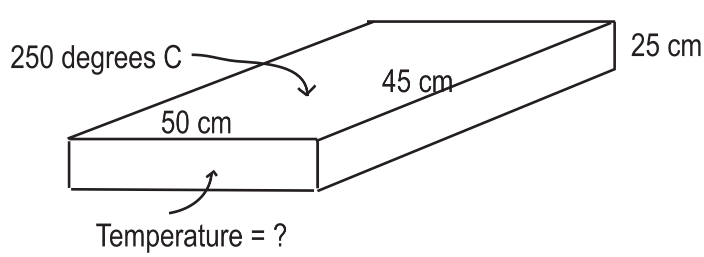
Examples:

The plate in the first example is made of carbon steel 2 cm thick, 300 W is lost by radiation. Calculate the inside temperature. Heat flux is constant at steady state at surface; we balance conduction with convection and radiation.

$$q_{\text{cond}} = q_{\text{conv}} + q_{\text{rad}}$$

$$-kA\frac{\Delta T}{\Delta x} = \left(2.156+0.3\right) = 2.456\ kW$$

$$\Delta T = \frac{2.456\ kW\left(0.2\,m\right)}{\left(0.5\,m\right)\left(0.75\,m\right)\left(43\frac{w}{mK}\right)}$$

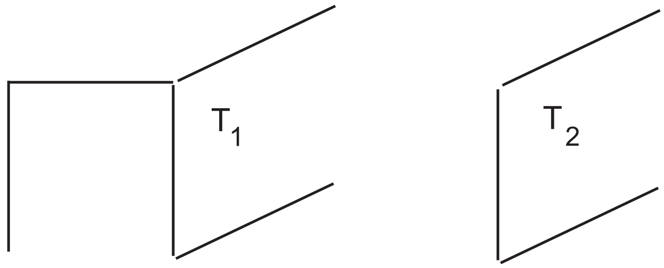


$$\Delta T = -3.05°\text{C}$$

The inside temperature

= 250 + 3.05 = 253.05°C

Radiation:



Two ∞ black plates at 800 and 300°C exchange heat by radiation. What is $q = Q/A$

$$Q/A = q = \sigma\left(T_1^4 - T_2^4\right)$$
$$= \left(5.665\cdot 10^{-8}\right)\left(1073^4 - 573^4\right)$$
$$= 69.03\ \text{kW/m}^3$$

Total Heat Loss by Convection and Radiation

Horizontal steel pipe 5 cm OD @ 50°C is enclosed in a 200°C room. $\in_{\text{pipe}} = 0.8$ What is heat loss in pipe?

Total loss = convection + radiation

From table $h = 6.5\ \text{W/m}^2\ K$ surface area $= \pi dL$ so

$$Q_{(\text{conv})}/L = h\left(\pi d\right)\left(T_w - T_\infty\right)$$
$$= \left(6.5\frac{\text{w}}{\text{m}^2}C\right)\left(3.14\right)\left(.05\ \text{m}\right)\left(50-20\right)$$
$$= 30.63\ \text{W/m}^2$$

Since the pipe is enclosed, all radiation hits walls so

$$Q_{(\text{rad})}/L = \in_1\left(\pi d\right)\sigma\left(T_w^4 - T_\infty^4\right)$$
$$= 25.04\ \text{W/m}$$
$$\text{Total} = \left(30.63 + 25.04\right) = 55.67\text{w/m}$$

and neglecting either is a mistake.

Heat Transfer — An Introduction

Review:
Thermal energy is transported by three mechanisms:

Mechanism	Origin of Transport	Math	Famous Guy
Conduction	Molecular Motion	$q = -k\frac{dT}{dx}$	Fourier
Radiation	Electromagnetic Waves	$q = \sigma T^4$	Stefan-Boltzmann
Convection	Motions of medium	$q = h(T - T_\infty)$	Newton

Terms

Q — heat flux units: $\frac{\text{energy}}{\text{time}}$

q — heat flux/area units: $\frac{\text{energy}}{\text{area-time}}$

T — temperature
k thermal conductivity, units: $\frac{\text{energy}}{\text{length-time-temperature}}$

h — heat transfer coefficient, units: $\frac{\text{energy}}{\text{area-time-temperature}}$

Heat transfer in which the primary mode is conduction is governed by Fourier's Law. In one dimension, this is expressed as

$$-q = -k\frac{dT}{dx}$$

The rate of heat transfer depends linearly on the temperature gradient and the thermal conductivity, *k*. The negative sign means that heat energy flows from hot to cold, as is generally observed. *k* is primarily a material parameter that reflects the resistance of the material to the transport of heat:

$$k \text{ for Air} \sim 0.15 \frac{\text{Btu}}{\text{ft hr °F}}$$

$$k \text{ for Silver} \sim 240 \frac{\text{Btu}}{\text{ft hr °F}}$$

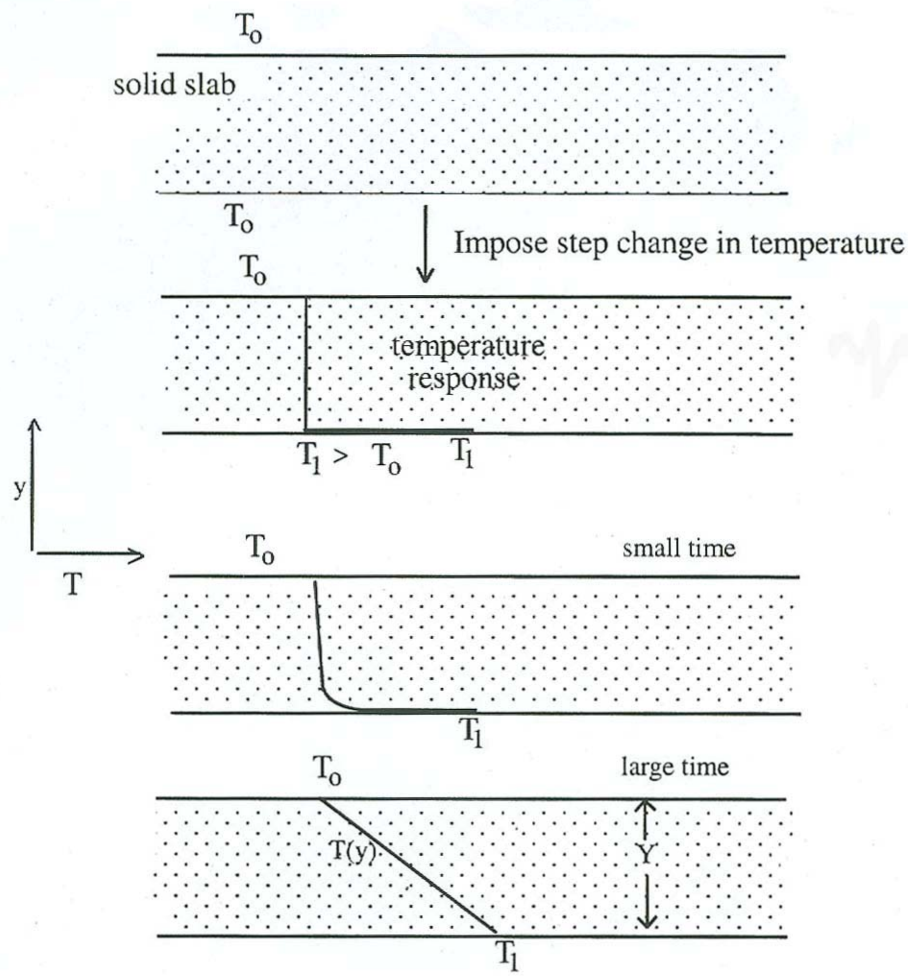
k varies by more than 5 orders of magnitude between materials. As in electrical conductivity, materials with high thermal conductivity are called conductors; materials with low thermal conductivity are called insulators.

Heat transfer in which the primary mode of transfer is due to fluid flow is called convection. If the fluid flow is induced by external forces, this is known as a forced convection. If the fluid flow is caused by temperature difference, it is known as natural convection. The heat transfer coefficient depends primarily on the fluid flow and fluid material properties.

Heat transfer in the absence of a medium, either solid, liquid, or vapor is radiation. Energy is transported via photons from one object to another that can be separated by a vacuum – this is how the sun heats the earth.

Thermal Conductivity – Where does it come from?
Imagine a uniform slab, initially at a constant uniform temperature, T_0. At some time, one side of the slab is exposed to temperature T_1. If Fourier's Law is valid for that material, the following will happen:

Fourier's Law



Experimentally, for small ΔT, at large times, simple materials react in a simple way

$$\frac{Q}{A} = q = -k\frac{\Delta T}{Y}$$

in the differential sense $q = -k\frac{dT}{dx}$, or the heat flux is linearly proportional to the thermal gradient. The constant of proportionality is the thermal conductivity. This is a "linear" model in that the heat flux is related to ΔT rather than $(\Delta T)^2$ or $(\Delta T)^3$.

In general for k
gases < insulators < ordinary liquids < non-metallic solids < liquid metals < alloys < pure metals.

Electrical conductivity and heat conductivity usually are similar —
high electrical conductivity $\rightarrow$ high thermal conductivity

For gases,	$k \uparrow$	with	$T \uparrow$	}	
	$k \uparrow$	with	$P \uparrow$	}	implies diffusion mechanism
	$k \downarrow$	with	$MW \uparrow$	}	
For metals, liquids	$k \downarrow$	with	$T \uparrow$	}	
	$k \downarrow$	with	$P \uparrow$	}	implies electron transport, phonons
	k ?	with	$MW \uparrow$	}	

<u>Conduction is the name for heat transfer in the absence of bulk motion or radiation</u>

In the completely general case, Fourier's Law is a tensorial relation

$$\frac{Q_i}{A_i} = \sum_{j=1}^{3} k_{ij} \frac{\partial T}{\partial x_j}$$

$$\frac{Q_1}{A_1} = k_{11} \frac{\partial T}{\partial x_1} + k_{12} \frac{\partial T}{\partial x_2} + k_{13} \frac{\partial T}{\partial x_3}$$

Most materials, however, are

<u>Orthotropic</u> $k_{ij} = k_i \delta_{ij} \rightarrow k_{ij} = 0$ for $i \neq j$
which means a gradient of temperature in the x direction causes a flow of heat in the x direction. Note that $k_{ii} \neq k_{jj}$ for all materials — wood, composites, some crystals have quite different thermal conductivities in different directions.

and/or
<u>Homogeneous</u> k_{ij} are independent of position in a material

and/or <u>Isotropic</u> $k_{ij} = k =$ constant independent of direction or position in a material
In virtually all chemical engineering applications, we assume k is isotropic; hence, independent of position, and as a consequence, orthotropic and homogeneous.

It can often be instructive to show how thermal conductivity can arise out of the simple motion of molecules. Thermal conductivity of monatomic gases at low density is a good example of how this occurs:

Assume:

1) Molecules are rigid non-interacting sphere of mass m and diameter *d*

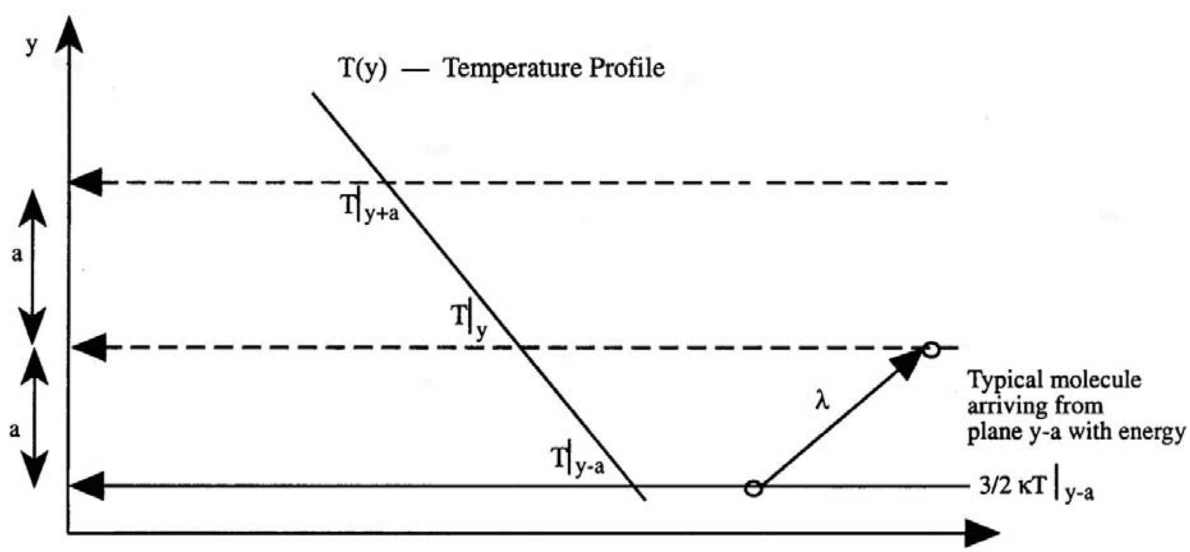
$\bar{u} = \left(\frac{8\kappa T}{\pi m}\right)$ — mean thermal velocity from gas kinetic theory

κ - Boltzmann's constant: 1.38×10^{-16} erg molecule^{-1} K^{-1}
m - molecular mass

$Z = 1/4\, n\bar{u}$ = plane collision frequency per unit area

$\lambda = \frac{1}{\sqrt{2}\pi d^2 n}$ mean free path of a molecule of diameter *d*

n = number density $\frac{\text{mol}}{\text{volume}}$



a - molecules reaching any plane are $2\lambda/3$ from last collision

2) For simple collisions, we will assume that only kinetic energy can be exchanged by collision.

From Gas Kinetic Theory

Average kinetic energy is given by Boltzmann's distribution:

$$\frac{1}{2}\mathrm{m}\,\bar{\mathrm{u}}^2 = 3/2\kappa T$$

The heat flux is just the different in thermal energy transported by the molecules as they cross the plane y from above and below

$$\text{Heat flux} = \begin{matrix}\text{energy flux}\\\text{from below}\end{matrix} - \begin{matrix}\text{energy flux}\\\text{from above}\end{matrix}$$

$$q_y = Z\left(1/2\,\mathrm{mu}^2\,|_{y-a}\right) \qquad -Z\left(1/2\,\mathrm{mu}^2\,|_{y+a}\right)$$

$$q_y = \frac{3}{2}\kappa Z\left(T_{y-a} - T_{y+a}\right)$$

If a is small (molecular dimensions) compared to the length scale for the thermal gradient (macroscopic), then we can expand differences in a Taylor series:

$$T\,|_{y-a} = T\,|_y - a\frac{dT}{dy} = T\,|_y - 2/3\lambda\frac{dT}{dy}$$

$$T\,|_{y+a} = T\,|_y + a\frac{dT}{dy} = T\,|_y + 2/3\lambda\frac{dT}{dy}$$

All molecules have a velocity equal to the place of first collision. Assume T_y does not vary too much over a few mean collision paths, then

$$qy = \frac{3}{2}\kappa Z\left[\left(T\,|_y - 2/3\lambda\frac{dT}{dy}\right) - \left(T_y + 2/3\lambda\frac{dT}{dy}\right)\right]$$

$$= -2\kappa Z\lambda\frac{dT}{dy} = -2\kappa\left(\frac{1}{4}\mathrm{n}\,\bar{\mathrm{u}}\right)\lambda\frac{dT}{dy}$$

$$qy = -\frac{1}{2}\mathrm{n}\,\kappa\,\bar{\mathrm{u}}\lambda\frac{dT}{dy}$$

We see that this is identical to Fourier's law if:

$$k = -1/2\,\mathrm{n}\,\kappa\,\bar{\mathrm{u}}\,\lambda;$$

Evaluating in terms of $\bar{\mathrm{u}}, \lambda$ we get

$$\boxed{k = \frac{1}{d^2}\sqrt{\frac{\kappa^3 \mathrm{T}}{\pi^3 \mathrm{m}}}}$$

This gives a reasonable estimate of k for simple gases. In fact, measurements of thermal conductivity and viscosity are often used to evaluate *d,* the effective molecular parameter for gases.

For condensed phases, molecular transport is not as important as other mechanisms of heat transfer.
For pure metals, thermal conductivity may be estimated (ChE handbook, pp 3-24).

$$k = \left(2.61 \text{ x } 10^{-8}\right) - \frac{T}{\rho_e} - \frac{\left(2 \text{ x } 10^{-17}\right)\left(\frac{T}{\rho_e}\right)^2}{C_p \rho} \frac{97 C_p \rho}{MT}$$

where k is in units of watts/cm − C, T is in °K, ρ_e is the electrical resistivity in ohm-cm, C_p is the heat capacity in cal/gr − °C, ρ is the density in g/cm^3 and M is the molecular weight in gr/mole.

For many varieties of wood, the thermal conductivity can be estimated by

$$k = \rho(0.1159 + .00233\, M_c) + .01375$$

where *k* is the thermal conductivity in Btu/hr/ft °F, ρ is the density in g/cm^3 and the M_c is the average moisture content.

For pure liquids, a simple correlation for *k* is

$$k = 2.80 \left(\frac{\tilde{N}}{\tilde{V}}\right)^{2/3} \kappa V_S$$

$$V_S = \sqrt{\frac{C_p}{C_v}\left(\frac{\partial p}{\partial \rho}\right)_T}$$

V_S is the velocity of sound in the liquid, κ is Boltzmann's constant and $\frac{\tilde{N}}{\tilde{V}}$ is the volume per molecule, which is equal to the molar volume divided by Avogadro's number. The quantity $\left(\frac{\partial p}{\partial \rho}\right)_T$ is related to the isothermal compressibility and is generally readily available.

Applications of Fourier's Law to Heat Transfer Calculations

Shell energy balances

Heat transfer is a more specific study of the general principal of energy conservation — we remember for a conserved quantity:

Input — Output = Accumulation

For interconversion of energy sources, we require the more general balance:

Input — Output + Generation — Consumption = Accumulation

As we are mostly concerned with discovering temperature distributions, we usually are dealing with thermal (as opposed to chemical, electrical, etc.) energy.

$\frac{\text{Thermal energy}}{\text{area}}$, q, can enter or leave a system by

— conduction $q = -k\frac{dT}{dx}$ — Fourier's Law **Remember this**

— convection $q = -k\left(T_S - T_f\right)$ — T_S — "surface" temperature } Newton's Law

T_f — "fluid" temperature }

— we will get to radiation later

As these equations are general, they apply to entire systems as well to differential elements within systems.

The size of an element can "mathematically" approach zero so as to form differential equations for the temperature distribution. To solve the differential equations, boundary conditions are necessary:

— Temperature at a point may be specified
— Heat flux, q, at a point may be specified
— Two types of heat flux might meet at a surface, conduction = convection

$$-k\frac{dT_S}{ds} = h\left(T_S - T_f\right)$$ **Remember this**

— Heat flux is generally continuous through interfaces
— Temperature is generally continuous through interfaces
— Symmetry

We will generally devise a geometric slab or shell so that it is perpendicular to the direction of heat flow. This simplifies the mathematical description immensely. The heat flux then becomes one dimensional.

In this scheme, we can also have sources of heat such as

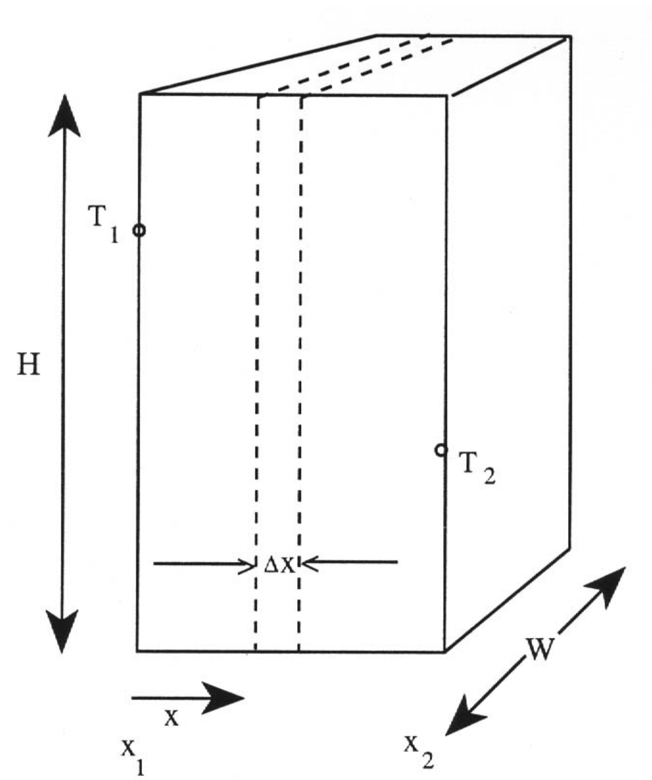
- Degradation (resistance) of electrical energy
- Nuclear fragments slowing down
- Viscous dissipation
- Conversion of chemical (or other internal energy) into heat

The Continuum Postulate

Anytime we create a differential equation, we assume that the material and or process is satisfactorily described as a continuum. That is, the temperature distribution, density, etc., are all smooth, differentiable functions. This description of the world is known as the continuum postulate. It is valid when the mean free path, or more generally, a characteristic molecular length scale, is small compared to the characteristic dimension of the system under study. We know that, if the differential length really did go to zero, we would be overwhelmed by the molecular, and even atomic nature of matter. However, when we let some differential length go to zero physically, what we really mean is that it becomes very small in comparison to the actual physical dimension of the system, but remains large compared to the molecular dimensions of a system. The continuum postulate works great unless the inhomogeneities in the system have a length scale that is appreciable to the overall scale of the system. In homogeneous solid slabs, bars, etc., the lengthscale of the inhomogeneities are microscopic and we are typically interested in the macroscopic temperature distribution, so that continuum postulate is fine. However, for gases at low density, the mean free path might be larger than the dimensions of the system, and the continuum postulate cannot be used.

Solutions to Fourier's Law: Insulating the house

One-dimensional heat conduction – wall or slab



Assume that we have a homogeneous slab of height H, width W, and thickness ($x_2 - x_1$). Assume also that H, W >> ($x_2 - x_1$) and that we impose that the temperature at $x_1 = T_1$, and at $x_2 = T_2$ for all time. Rewriting in terms of the steady-state energy flux, *q*, entering a small slab of width Δx gives

Input – output = 0

Our energy balance gives at steady state:

energy into slab		energy leaving slab
$q\|_x \, HW$	–	$q\|_{x+\Delta x} \, HW = 0$

Rearranging to put the above into a more familiar form:

$$\left(\frac{q|_{x+\Delta x} - q|_x}{\Delta x}\right) HW \; \Delta x = 0; \qquad \lim_{\Delta x \to 0} \frac{q|_{x+\Delta x} - q|_x}{\Delta x} = 0 = \frac{dq_x}{dx}$$

The solution to this differential equation is simply:

q = constant = q_0. **Remember this**

This says that the heat flux through a thin, homogeneous slab is constant at steady state.

Now, we need relationships between *q* and *T*. So far no models have been imposed for the material. Here we will assume the material is well represented by a homogeneous case of Fourier's law:

$$q = -k\frac{dT}{dx}$$

We can separate this differential equation and integrate:

$$-\frac{q_0}{k}dx = dT; \quad -\frac{q_0}{k}\int dx = \int dT \text{ for constant } k$$

to get: $T = -\frac{q_0}{k}x + C_2; C_2, q_0$ are integration constants. We now need to impose the boundary conditions: i.e., that the temperatures at x_1 and x_2 are known to be T_1 and T_2 respectively:

$T = T_1$ @ $x = x_1$, $\quad T = T_2$ @ $x = x_2$

at x_2 we make sure that $T = T_2$

a) $\quad T_2 = -\frac{q_0}{k}x_2 + C_2$

at x_1 we make sure that $T = T_1$

b) $\quad -\left(T_1 = -\frac{q_0}{k}x_1 + C_2\right)$

If we add these two together, we can get rid of C_2

c) $\quad T_2 - T_1 = -\frac{q_0}{k}(x_2 - x_1)$; and, solving for q_0 we get:

d) $\quad q_0 = k\frac{T_2 - T_1}{x_2 - x_1}$ — heat flux — constant. We can now see that for a simple one dimensional slab:

$q_0 = k\frac{\Delta T}{\Delta x}$ **<u>Really remember this as it is the basis for insulation</u>**

Inserting this value for q_0 in Equation (a) we also can get the temperature distribution in terms of easy to measure quantities:

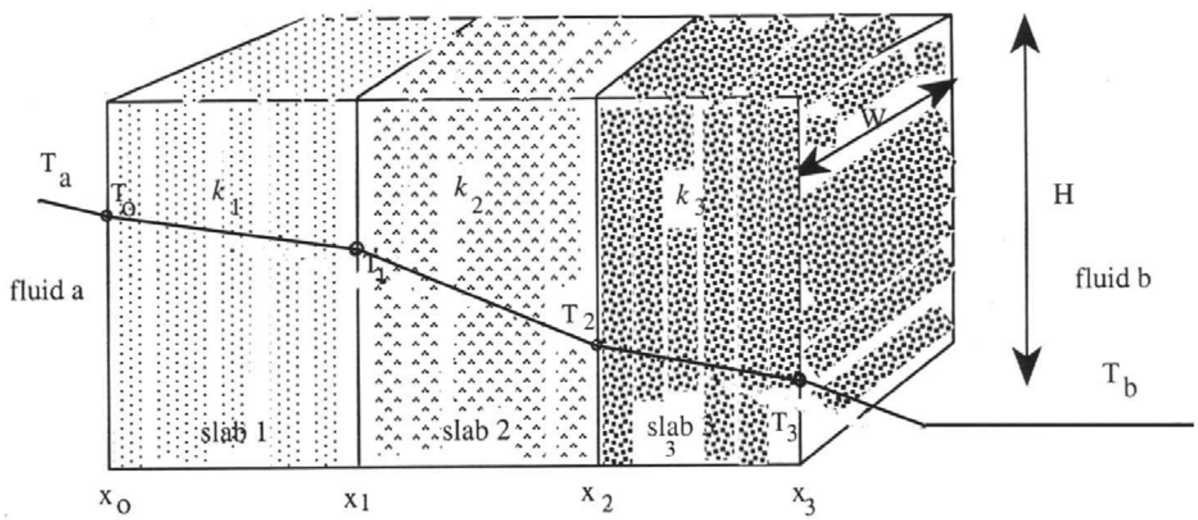
<u>Remember this</u>

$T = T_2 - \frac{T_2 - T_1}{x_2 - x_1}(x_2 - x)$ — and the steady state temperature distribution in a simple slab is linear. Note that the temperature distribution is independent of *k,* hence the material in the slab. However, the heat flux, *q,* depends on *k.* This always has surprised me, but without this result, it would be tough to understand insulation. This is one of those equilibrium results that is independent of the details.

Philosophical Motivation: The general point to most mathematical analysis of heat transfer problems is to describe things that are hard to measure directly (like heat flux or the temperature inside a wall) in terms of things that are easier to measure, like the temperatures at the outside of the wall. As long as you keep this in mind, heat transfer problems will make a lot more sense.

Combined Convective and Conductive Heat Transfer

Composite Wall with Convection



This example forms the basis of the insulation industry, which employs thousands of hardworking Americans. Imagine we have a wall consisting of several layers of different materials in contact with fluids on either side.

The heat flux through each slab is constant as we showed in the previous example. When a slab is in contact with a fluid, we remember that heat transfer is given by Newton's Law:

$h_a(T_a - T_0) = \frac{q}{WH} = h_b(T_3 - T_b)$ $T_a - T_b$ — temperature of fluids far from wall on sides a and b.

While within each slab, we know from the previous example:

$\frac{q}{A} = -k\left(\frac{dT}{dx}\right)$ is constant and

$$\frac{q_1}{WH} = \frac{k_1(T_0 - T_1)}{x_1 - x_0} = \frac{q_2}{WH} = \frac{k_2(T_1 - T_2)}{x_2 - x_1} = \frac{q_3}{WH} = \frac{k_3(T_2 - T_3)}{x_3 - x_2} = q$$

The heat flux is the same in each slab. If it weren't, there would be accumulation of energy, which can't happen at steady-state.

We can manipulate these equations to eliminate things that are hard to measure:

$$T_a - T_0 = \frac{q}{h_a}$$

$$T_0 - T_1 = \frac{q(x_1 - x_0)}{k_1}$$

$$T_1 - T_2 = \frac{q(x_2 - x_1)}{k_2}$$

$$T_2 - T_3 = \frac{q(x_3 - x_2)}{k_3}$$

$$T_3 - T_b = \frac{q}{h_b}$$

→ By adding up all of these equations, we get rid of needing to know all the intermediate temperatures, which would be hard to measure (unless we drilled a bunch of holes in the wall.) Or, if we know *q*, we could determine T.

$$T_a - T_b = q\left[\frac{1}{h_a} + \frac{\Delta x_1}{k_1} + \frac{\Delta x_2}{k_2} + \frac{\Delta x_3}{k_3} + \frac{1}{h_b}\right]$$

or

$$\boxed{q = \frac{T_a - T_b}{\frac{1}{h_a} + \sum_i \frac{\Delta x_i}{k_i} + \frac{1}{h_b}}}$$

This gives the heat flux through a composite wall as a function of only the inner and outer fluid temperatures and the wall material and thickness.

This can be written as follows:

$$q = U(T_a - T_b) \quad U = \left[\frac{1}{h_a} + \sum_i \frac{\Delta x_i}{k_i} + \frac{1}{h_b}\right]^{-1}$$

U is the overall heat transfer coefficient.

Total flux is qA or:

$$\boxed{Q = UA(T_a - T_b)}$$

→ In analogy to Ohm's law in electricity, we see that heat flow is proportional to a driving force, (T_a - T_b), and the material resistance, UA. This is the basic design equation for insulation. Once we know *Q* or *q*, we can determine T within the wall.

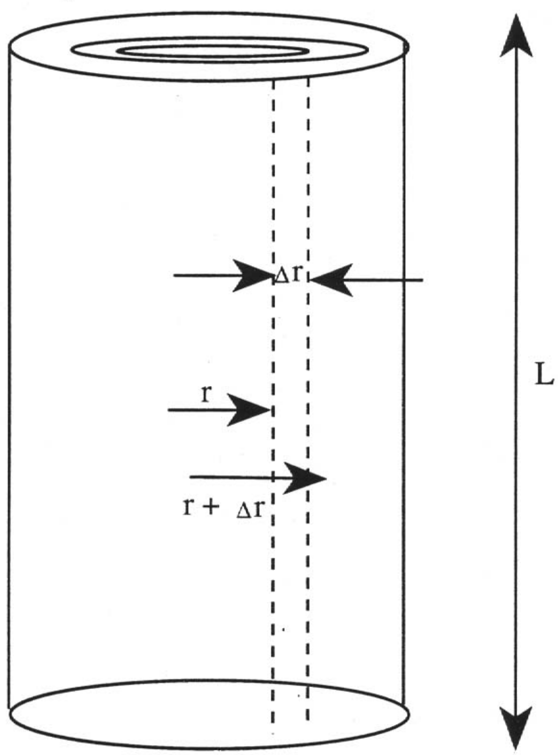
Cylindrical Geometry — Complications Due to Changing Area

Consider an electrical wire carrying current — radius R, current density I. The rate of heat production/volume in the wire is just the amount of energy lost due to the resistance of the wire

$S_e = I^2/k_e$ $\qquad$ k_e — electrical conductivity

$$k_e \, \alpha \, \frac{1}{\text{Resistance}}$$

The temperature of the surface of the wire can be assumed constant @ T_0



If the wire is much longer than it is thick, then we can assume heat is being transferred in the radial direction only. (This assumption depends on what we want to know as well as the boundary conditions.)

Our balance is in the r direction. Consider an energy balance in the cylindrical shell between r and r + Δr (hence, the cylindrical shell is always perpendicular to the heat flow).
Input to shell — output to shell + generation in shell
— consumption in shell = accumulation in shell.

if q_r is the heat flux/area in the r direction, we have

$$\underbrace{(\text{heat flux @ } r \cdot \text{area for flux @ } r)}_{\text{Input}} - \underbrace{(\text{heat flux @ } r + \Delta r \cdot \text{area @ } r + \Delta r)}_{\text{Output}}$$

$$+ \underbrace{\left(\frac{\text{source}}{\text{volume}} \cdot \text{volume of shell}\right)}_{\text{generation}} = \text{Accumulation} = 0 \text{ @ steady state}$$

or in mathematical terms:

(A) $\{2\pi\ L\ (r\ q_r)|_r\} - \{2\pi\ L\ (r + \Delta r)\ q_r|_{r+\Delta r}\}$
$+\{2\pi\ r\ \Delta r\ L\}S_e=0$

Digression

(volume of cylindrical shell) = $(\pi\ (r + \Delta r)^2\ L - \pi\ r^2\ L) =$

$= \pi\ r^2\ L + (2\pi\ r\ \Delta r)\ L + \pi\Delta r^2\ L) - \pi r^2\ L = 2\pi\ r\ \Delta r\ L + \pi\ \Delta r^2\ L$

To terms linear in Δr, this is $2\pi\ r\Delta r\ L$ (we neglect terms in $(\Delta r)^2$).

Rearranging (A), we get

$$2\pi\ L\left\{\frac{(rq_r)|_{r+\Delta r} - (rq_r)|_r}{\Delta r}\right\} - 2\pi\ Lr\ S_e = 0 \text{ (Note that } (r + \Delta r) = r|_{r+\Delta r})$$

If we take $\Delta r \to 0$, the first term is the r derivative of (rq_r) and we get (sometimes students divide by the r term, but when we keep the r + Δr notation, these are clearly different values of r)

$$\left\{\frac{d}{dr}(rq_r)\right\} = rS_e$$

This can be integrated (for S_e constant):

$$\int d(rq_r) = \int rS_e\ dr$$

$$rq_r = \frac{r^2 S_e}{2} + C_1$$

$$q_r = \frac{r\,S_e}{2} + \frac{C_1}{r}$$

We now have an expression for the energy flux. We can say $C_1 = 0$ so as to keep the flux finite @ r = 0. (This is a standard physical constraint to put on the mathematics. We know we can't have a physical infinity of heat flux in a wire, so we have to get rid of these terms somehow.)

Digression: Symmetry along the wire axis also gives us a boundary condition@ r = 0. If the wire is axially symmetric $T_{+r} = T_{-r}$ near r = 0 or $T_{\Delta r} = T_{-\Delta r}$, where Δr is a small number. Hence

$$\lim_{\Delta r=0} \frac{T_{\Delta r} - T_{-\Delta r}}{2\Delta r} = 0 \quad \text{or} \quad \frac{\partial T}{\partial r}\Big|_{r=0} = 0.$$

This can be used at any point, line, or plane of symmetry. In this example

$$\frac{dT}{dr}\Big|_{r=0} = 0 = \frac{0 \cdot S_e}{2} + \frac{C_1}{0} \text{ or } C_1 = 0 \quad \text{as before.}$$

Inserting the appropriate form of Fourier's Law to relate the heat flux to the temperature distribution

$$q_r = \frac{rS_e}{2} = -k\frac{dT}{dr}$$

We can integrate again to get

$$T = \frac{r^2 S_e}{4k} + C_2$$

We can now impose the final boundary condition to make sure T = T_0 at the surface of the wire.

$@r = R, \; T = T_0,$ so,

$$T_0 = \frac{S_e R^2}{4k} + C_2; \quad C_2 = T_0 + \frac{S_e R^2}{4k}$$

and the temperature distribution in a cylindrical wire is:

$$T - T_0 = \frac{S_e R^2}{4k}\left(1 - \left(\frac{r}{R}\right)^2\right)$$

Once the temperature distribution is known, other important quantities can be calculated: For instance, we can find:
Maximum temperature in the wire and its location

$\frac{dT}{dr} = 0 = \frac{S_e r}{2k} \rightarrow r = 0$ is the place of maximum temperature. This is a consequence of the symmetry of the wire. **<u>Remember this</u>**

$T_{max} - T_0 = \frac{S_e R^2}{4k}; T_{max} = T_0 + \frac{S_e R^2}{4k}$ is the maximum temperature. This might be very important for design purposes to insure the wire doesn't melt.

We can also calculate an average temperature. For a cylindrical geometry, the average is:

$$\left\langle T - T_0 \right\rangle \equiv \frac{\int_0^{2\pi}\int_0^R \left(T(r) - T_0\right) r dr d\theta}{\int_0^{2\pi}\int_0^R r dr d\theta}$$

<u>Remember this</u>

$$= \frac{S_e R^2}{8k} \left\langle T \right\rangle = T_0 + \frac{S_e R^2}{8k}, \text{ and } \left(T_{max} - T_0\right) = 2\left(T_{av} - T_0\right)$$

We can also calculate the total heat flow at surface

$$Q|_r = 2\pi RL \cdot q|_r = R$$

$$= 2\pi RL \cdot \frac{S_e R}{2}$$

$$= \pi R^2 L \cdot S_e$$

which is identical to the total heat generated by the wire due to electrical resistance.

Which also just means that the heat generated better leave through the walls of the wire, if we don't want to accumulate energy.

What if our wire was in contact with air. Instead of a fixed surface temperature, a more realistic boundary condition is to use Newton's law of convection:
@ r = R   conduction in wire = convection to air

$$-k\ dT/dr\,|_{r=R} = h\left(T\,|_{r=R} - T_{\text{air}}\right)$$

@ r = 0 $\quad \dfrac{dT}{dr} = 0 \qquad$ by symmetry

Our general solution from before is unchanged, just C_2 changes

$$T_{(r)} = C_2 - \frac{S_e r^2}{4k}, \quad T(R) = C_2 - \frac{S_e R^2}{4k}$$

$$\frac{\partial T}{\partial r}\Big|_{r=R} = -\frac{S_e R}{2k}, \quad -k\frac{dT}{dr}\Big|_{r=R} = \frac{+S_e R}{2}$$

so

$$\frac{S_e R}{2} = h\left(\left(C_2 - \frac{S_e R^2}{4k}\right) - T_{\text{air}}\right)$$

And

$$\frac{S_e R}{2h} + \frac{S_e R^2}{4k} + T_{\text{air}} = C_2 \quad \text{and}$$

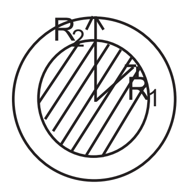
$$T_{(r)} - T_{\text{air}} = \frac{S_e R^2}{4k}\left(1 - \frac{r^2}{R^2}\right) + \frac{S_e R}{2h}$$

and the temperature @ $r = R$ is

$$T(R) = T_{\text{air}} + \frac{S_e R}{2h} = T_0 \text{ from previous example}$$

$$\text{and } T_{\max} = T_{\text{air}} + \frac{S_e R}{2h} + \frac{S_e R^2}{4k}$$

How about an insulated wire? We have the current carrying wire inside a plastic coating. The wire has a radius of R_1 and thermal conductivity k_1. The plastic carries no current $(S_e = 0)$, is of radius R_2 and thermal conductivity k_2.



In the wire, we will still have $\dfrac{\partial T}{\partial r}\Big|_{r=0} = 0$ so then for $0 \le r \le R_1, \quad T(r) = C_2 - \dfrac{S_e R_1^2}{2k_1}$

but there is no heat generation in the plastic; so

$$\text{for } R_1 \le r \le R_2 \quad \frac{d}{dr}\left(r q_r\right) = 0 \quad (S_e = 0)$$

$$r q_r = C, \quad q_r = C/r$$

We can't say anything obvious about C since $R_1 \le r \le R_2$

We can say that the heat flux and the temperature should be continuous at the wire-plastic boundary:

$$q_{r=R_1}^{\text{wire}} = q_{r=R_1}^{\text{plastic}}$$

$$\frac{S_e R_1}{2} = \frac{C}{R_1}; \quad C = \frac{S_e R_1^2}{2},$$

$$q_r^{\text{plastic}} = \frac{S_e R_1^2}{2r} = -k_2 \frac{dT}{dr}$$

$$\frac{dT}{dr} = -\frac{S_e R_1^2}{2k_2 r}$$

$$\int dT = \int -\frac{S_e R_1^2}{2k} \frac{dr}{r}$$

$$R_1 \le r \le R_2 \quad T(r) = -\frac{S_e R_1^2}{2k_2} \ell nr + C_3$$

The temperature must also be continuous @ $r = R$,

$$-\frac{S_e R_1^2}{2k_2} \ell n\, R_1 + C_3 = \left(-\frac{S_e R_1^2}{2k_1} + C_2 \right)$$

And if we use $T = T_0$ @ $r = R_2$ as the final condition

$$-\frac{S_e R_1^2}{2k_2} \ell n\, R_2 + C_3 = T_0$$

$$C_3 = T_0 + \frac{S_e R_1^2}{2k_2} \ell n\, R_2$$

so

$$T_0 + \frac{S_e R_1^2}{2k_2} \ell n\, R_2 / R_1 - \frac{S_e R_1^2}{4k_1} + C_2$$

$$C_2 = T_0 + \frac{S_e R_1^2}{2k_2} \ell n\, R_2 / R_1 = \frac{S_e R_1^2}{4k_1}$$

And we have

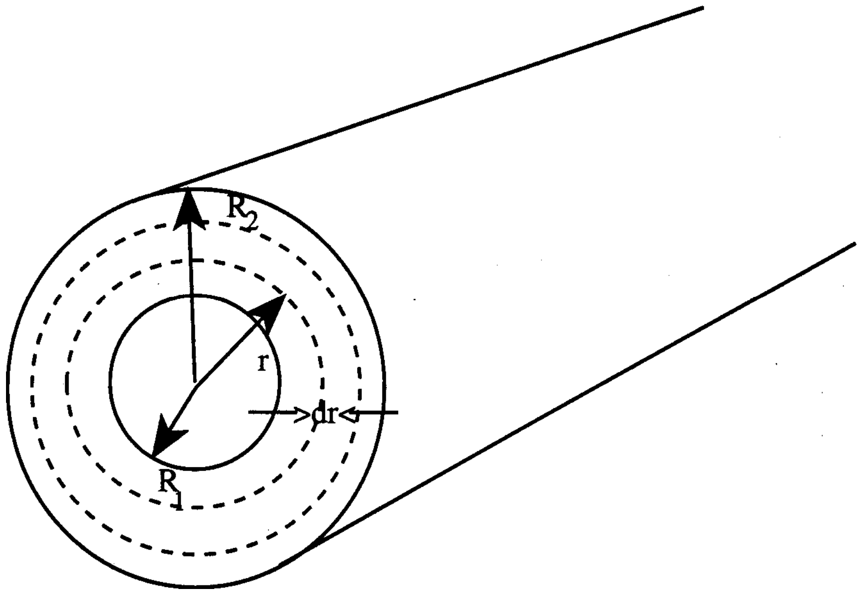
$$O \le r \le R_{1,} \quad T(r) = T_0 + \frac{S_e R_1^2}{2k_2} \ell n\, R_2 / R_1 + \frac{S_e R_1^2}{4k_1} \left(1 - \frac{r^2}{R_1^2} \right)$$

$$R_1 \le r \le R_{2,} \quad T(r) = T_0 - \frac{S_e R_1^2}{2k_2} \ell n \;\; r / R_2$$

So as we add details to our model wire, our solution grows in complexity.

Composite Cylinders, or Insulating Pipes

We need to re-derive the relevant insulation equation for a cylindrical geometry, similar to the slabs we worked on before.



The steady state flow of heat in a bar of cylindrical cross section with no source.

Macroscopic Balance Equation:

Input - Output = Accumulation = 0

(Flux . area) $|_r$ - (Flux · area) $|_{r+\Delta r} = 0$

$$q|_r = 2\pi \, r \, L|_r - \cdot q_r \;\; 2\pi \, r \, L|_{r+\Delta r} = 0$$

q_r is the heat flux per unit area. Dividing by Δr, we get

$$\frac{(rq_r)|_{r+\Delta r} - (rq_r)|_r}{\Delta r} = 0$$

in the limit as $\Delta r \to 0$ this becomes

$$\frac{d}{dr}\left(rq_r\right) = 0$$

Integrating once we see

$$rq_r = C_1$$

$$q_r = C_1 / r$$

In our composite cylinder, r doesn't go to zero, so we have to evaluate C_1 later.
Now, we use the appropriate form of Fourier's Law to relate the heat flux to the temperature.

$$q_r = -k\frac{dT}{dr}$$

$$-k\frac{dT}{dr} = \frac{C_1}{r}d$$

$$\frac{dT}{dr} = -\frac{C_1}{kr}$$

Integrating again, we get

$$T(r) = -\frac{C_1}{k}\ell n r + C_2$$

Applying the boundary conditions

$$T = T_1 \text{ at } r = r_1$$

$$T = T_2 \text{ at } r = r_2$$

$$T_1 = -\frac{C_1}{k}\ell n r_1 + C_2$$

$$T_2 = -\frac{C_1}{k}\ell n r_2 + C_2$$

$$T_1\text{-}T_2 = +\frac{C_1}{k}\ell n \cdot \frac{r_2}{r_1}$$

Therefore,

$$C_1 = \frac{k(T_1 - T_2)}{\ell n(r_2/r_1)}$$

$$T(r) = -\frac{(T_1 - T_2)}{\ell n(r_2/r_1)}\ell n r + C_2$$

We now impose the boundary condition again:

$$T_1 = -\frac{-(T_1 - T_2)}{\ell n\, r_2/r_1}\ell n r + C_2$$

$$C_2 = T_1 + \frac{(T_1 - T_2)}{\ell n(r_2/r_1)}\ell n r_1$$

$$\boxed{T(r) = T_1 - \frac{(T_1 - T_2)}{\ell n(r_2/r_1)}\ell n(r/r_1)}$$

and from the temperature distribution, we can use Fourier's Law to determine the heat flux:

$$q_r = \frac{C_1}{r} = \frac{k(T_1 - T_2)}{r\ell n(r_2/r_1)},$$ and the local heat flux is not constant

However, we see that the total heat flux Aq_r is constant :

$$Aq_r = 2\pi\, r\, L\, q_r = 2\pi\, L\, \frac{k(T_1 - T_2)}{\ell n(r_2/r_1)}$$

In analogy to heat conduction through multiple walls, we can determine the insulation equations for cylindrical surfaces:

$$Q = \frac{2\pi\, Lk(r_2 - r_1)}{\ell n(r_2/r_1)}\frac{(T_1 - T_2)}{(r_2 - r_1)} = 2\pi\, L\frac{k(T_1 - T_2)}{\ell n(r_2/r_1)}$$

$$= \frac{(A_2 - A_1)}{\ell n(A_2/A_1)}\frac{\Delta T}{\Delta r}$$ (so that the equations resemble those for a flat wall)

$$Q = kA_{\ell m}\frac{\Delta T}{\Delta r}$$

A is the "log-mean" area for transfer of heat. Why did we go to this trouble to take a simple formula (above) into apparently a more complex one?

<u>Multilayer Cylinder</u> — Compare to the composite wall

As for a flat wall:

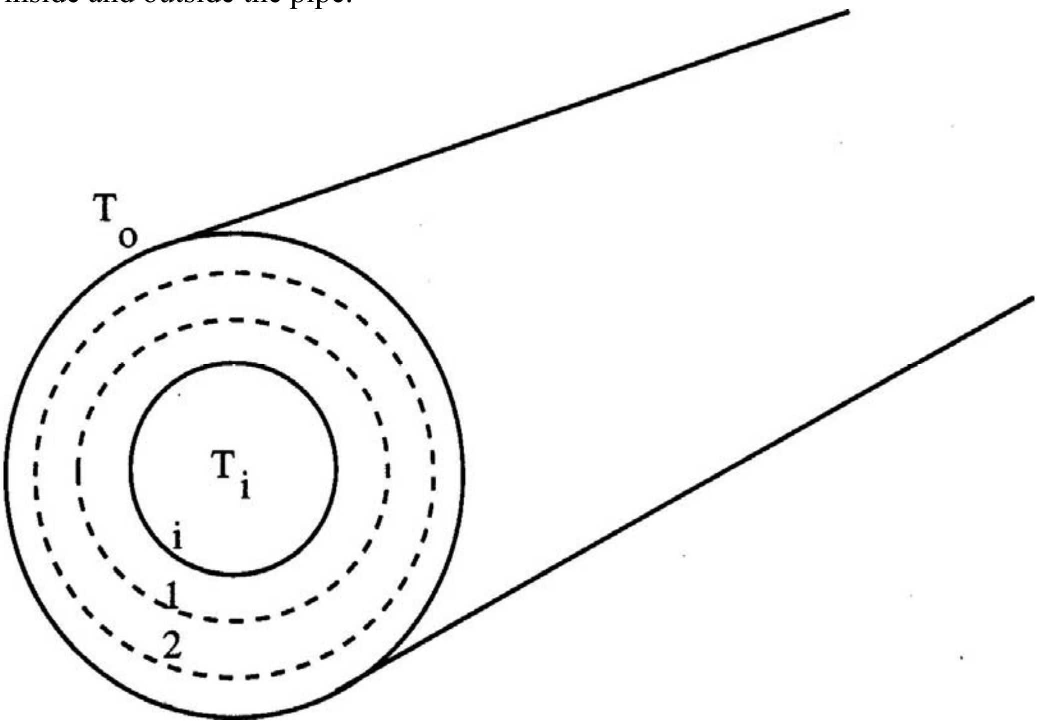
$$\text{Flow} = \frac{\text{Potential difference}}{\text{Resistance}} = \frac{\Delta T}{\Delta r/kA_{\ell m}}$$

for a composite cylinder of 3 layers a, b, c we know that the total heat flux is constant:

$$Q = \left(kA_{\ell m}\frac{\Delta T}{\Delta r}\right)_a = \left(kA_{\ell m}\frac{\Delta T}{\Delta r}\right)_b = \left(kA_{\ell m}\frac{\Delta T}{\Delta r}\right)_c$$

$$Q = \frac{\sum_i^n \Delta T}{\sum_i^n \left(\frac{\Delta r}{kA_{\ell m}}\right)_i}$$

We get an analogous result for pipes as for slabs when we consider convection to the fluid inside and outside the pipe.



$$T_i - T_0 = \frac{Q}{h_i A_i}$$

$$T_0 - T_1 = Q\frac{\Delta r_1}{k_1\ A_1\ \ell_m}$$

$$T_1 - T_2 = Q\frac{\Delta r_2}{k_2\ A_2\ \ell_m}$$

$$T_3 - T_0 = \frac{Q}{h_0\ A_0}$$

$$T_i - T_0 = Q\left[\frac{1}{h_i\ A_i} + \sum_j \frac{\Delta r_j}{k_j\ A_{j\ell m}} + \frac{1}{h_0\ A_0}\right]$$

and

$$Q = \left[\frac{T_i - T_0}{\frac{1}{h_i\ A_i} + \sum_j \frac{\Delta r_j}{k_j\ A_{j\ell m}} + \frac{1}{h_0\ A_0}}\right]$$

For a cylinder, we can base our overall heat transfer coefficient on either the inside or outside area of the cylinder:

$Q = A_i U_i \left(T_i - T_0\right)$ or $Q = U_0 A_0 \left(T_i - T_0\right)$

U_0 — outside overall heat transfer coefficient

$$Q = \frac{A_0(T_i - T_0)}{\frac{A_0}{h_i A_i} + \sum_j \frac{\Delta r_j A_0}{k_j A_{j\ell m}} + \frac{1}{h_0}} = U_0 A_0 (T_i - T_0); \frac{1}{U_0} = \frac{A_0}{h_i A_i} + \sum_j \frac{\Delta r_j A_0}{k_j A_{j\ell m}} + \frac{1}{h_0}$$

$$Q = \frac{A_i(T_i - T_0)}{\frac{1}{h_i} + \sum_j \frac{A_i \Delta r_j}{k_j A_{j\ell m}} + \frac{1}{h_0}} = U_i A_i (T_i - T_0); \frac{1}{U} = \frac{1}{h_i} + \sum_j \frac{\Delta r_j A_i}{k_j A_{j\ell m}} + \frac{1}{A_0 h_0}$$

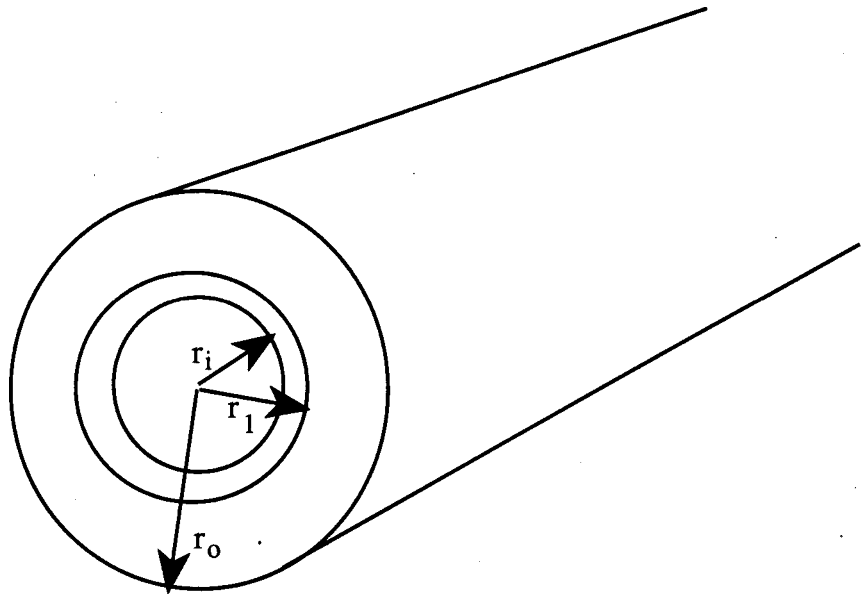
Note in simplification: If the change in areas going from the inside of the pipe to the outside is less than a factor of about 2, $U_0 \approx U_i \approx \frac{1}{h_i} + \sum_j \frac{\Delta r_j}{k_j} + \frac{1}{h_0}$ because $A_0 \approx A_i \approx A_{j\ell m}$.

In many engineering applications, h_i , h_0 are only known to ±50% so the log mean area often doesn't matter.

Critical Insulation Thickness

Question: Does adding insulation to a pipe always decrease heat losses? We begin from the general expression for heat flux from a cylinder:

$$Q = U_0 A_0 (T_0 - T_i)$$



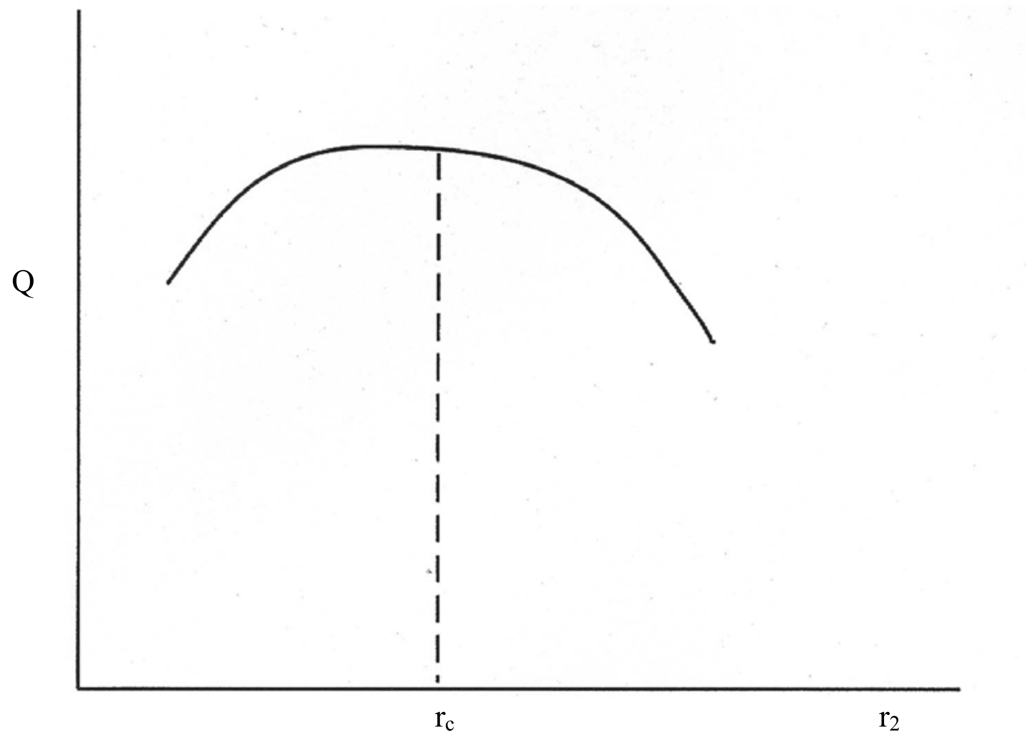
If we assume that most of the resistance is in the insulation layer from r_1 to r_0, we can let

$h_i \to \infty, \frac{k_1}{r_1} \to \infty$, and our result is simplified.

Then, in terms of r's (writing as before without $A_{j\ell m}$'s)

$$Q = \frac{2\pi r_0 L (T_i - T_0)}{\frac{r_0}{k_0} \ell n \frac{r_0}{r_1} + \frac{1}{h_0}}$$

if we graph Q vs r_0, the insulation thickness, then



There will always exist a critical pipe radius, beyond which the insulation will help.

To find r_c we take

$$\frac{\partial Q}{\partial r_0} = 0 = \frac{\left[\frac{r_0}{k_0}\ell n\frac{r_0}{r_1}+\frac{1}{h_0}\right]^2 - r_0\left[\frac{1}{k_0}\ell n\frac{r_0}{r_1}+\frac{1}{k_0}\right]}{\left[\frac{r_0}{k_0}\ell n\frac{r_0}{r_1}+\frac{1}{h_0}\right]^2}$$

solving for $r_0 = r_c$, which is the critical radius for which heat flux is a maximum:

$r_c = k_0 / h_0$

for good insulation k_0 = .035 Btu/hr ft °F

h_0 for still air — 2 Btu/hr ft °F

so r_c = 0.2 "

So for pipes larger than about 1/2 diameter, insulation will decrease heat loss. Of course, this doesn't really account for all of the costs associated with the insulation. In general, hot water pipes less than about 3 inches in diameter are not insulated, due to the expense of installing and maintaining the insulation.